

BEST CONSTANTS FOR A FAMILY OF CARLESON SEQUENCES

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ABSTRACT. We consider a general family of Carleson sequences associated with dyadic A_2 weights and find sharp – or, in one case, simply best known – upper and lower bounds for their Carleson norms in terms of the A_2 -characteristic of the weight. The results obtained make precise and significantly generalize earlier estimates by Wittwer, Vasyunin, Beznosova, and others. We also record several corollaries, one of which is a range of new characterizations of dyadic A_2 . Particular emphasis is placed on the relationship between sharp constants and optimizing sequences of weights; in most cases explicit optimizers are constructed. Our main estimates arise as consequences of the exact expressions, or explicit bounds, for the Bellman functions for the problem, and the paper contains a measure of Bellman-function innovation.

1. PRELIMINARIES

We will be concerned with weights on \mathbb{R} , i.e. locally integrable functions that are positive almost everywhere. Our weights will be assumed to belong to the dyadic Muckenhoupt class A_2^d associated with a particular lattice \mathcal{D} , i.e., the set of all dyadic intervals on the line uniquely determined by the position and size of the root interval. If an interval I is fixed, $\mathcal{D}(I)$ will stand for the unique dyadic lattice on I and $\mathcal{D}_n(I)$ for the set of the dyadic subintervals of I of the n -th generation, $\mathcal{D}_n(I) = \{J : J \in \mathcal{D}(I), |J| = 2^{-n}|I|\}$.

Let $\langle w \rangle_J$ be the average of a weight w over an interval J , $\langle w \rangle_J = \frac{1}{|J|} \int_J w$. A weight w is said to belong to A_2^d , written $w \in A_2^d$, if

$$[w]_{A_2^d} \stackrel{\text{def}}{=} \sup_{J \in \mathcal{D}} \langle w \rangle_J \langle w^{-1} \rangle_J < \infty.$$

The quantity $[w]_{A_2^d}$ is referred to as the A_2^d -characteristic of w . Observe that $[w]_{A_2^d} \geq 1$ and $w \in A_2^d$ if and only if $w^{-1} \in A_2^d$, in which case $[w]_{A_2^d} = [w^{-1}]_{A_2^d}$. For a number $Q \geq 1$, the set of all A_2^d weights w with $[w]_{A_2^d} \leq Q$ will be denoted by $A_2^{d,Q}$. If I is an interval and the supremum in the above definition is taken over all $J \in \mathcal{D}(I)$ instead of all $J \in \mathcal{D}$, we will write $A_2^d(I)$ and $A_2^{d,Q}(I)$, as appropriate.

A non-negative sequence $\{c_J\}_{J \in \mathcal{D}}$ is called a Carleson sequence if for all $I \in \mathcal{D}$

$$(1.1) \quad \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J \leq C < \infty.$$

The smallest such C , denoted by $\|\{c_J\}\|_C$, is called the Carleson norm of $\{c_J\}$. The significance of such sequences in analysis stems mainly from their role in the Carleson embedding theorem and related results. A key example is the following lemma whose proof can be found in [9] (in a more general, weighted setting). We will use this lemma to derive an important corollary of our main estimates.

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Lemma 1.1 (Carleson Lemma). *A sequence $\{c_J\}_{J \in \mathcal{D}}$ is a Carleson sequence with norm B if and only if for all non-negative, measurable functions F on the line,*

$$\sum_{J \in \mathcal{D}} c_J \inf_{x \in J} F(x) \leq B \int_{\mathbb{R}} F(x) dx.$$

This inequality suggests that it may be important to have good estimates of the Carleson norms of sequences one uses. One specific context where such need arises is when studying dyadic paraproducts in weighted settings. For example, in [1] and [9] the authors estimate the norms of paraproducts on $L^2(w)$, with $w \in A_2^d$. In this situation, the sequences of interest are often of the form

$$c_J^{(\alpha)}(w) \stackrel{\text{def}}{=} |J| \langle w \rangle_J^\alpha \langle w^{-1} \rangle_J^\alpha \left[\frac{(\Delta_J w)^2}{\langle w \rangle_J^2} + \frac{(\Delta_J w^{-1})^2}{\langle w^{-1} \rangle_J^2} \right],$$

where $\Delta_J(\cdot) = \langle \cdot \rangle_{J^-} - \langle \cdot \rangle_{J^+}$, and J^\pm are the two halves of J . In [1], Beznosova showed that the norm of $\{c_J^{(1/4)}(w)\}$ is no greater than $C[w]_{A_2^d}^{1/4}$ for a numerical constant C . In [13], Nazarov and Volberg extended this estimate, proving that

$$(1.2) \quad \|\{c_J^{(\alpha)}(w)\}\|_C \leq \frac{C}{\alpha - 2\alpha^2} [w]_{A_2^d}^\alpha, \quad \alpha \in (0, 1/2).$$

In a recent paper [9], Moraes and Pereyra observed that one can simply combine Beznosova's result with the fact that $t \mapsto t^\alpha$ is an increasing function for $\alpha > 0$ to obtain

$$\|\{c_J^{(\alpha)}(w)\}\|_C \leq r(\alpha) [w]_{A_2^d}^\alpha, \quad \alpha > 0,$$

for an explicit $r(\alpha)$.

The case $\alpha = 0$ was considered by Wittwer in [20], where she obtained the asymptotically sharp estimate

$$(1.3) \quad \|\{a_J(w)\}\|_C \leq 8 \log[w]_{A_2^d},$$

where $\{a_J(w)\}$ is the same as $\{c_J^{(0)}(w)\}$, except with only *one* of the two summands (it does not matter which one is removed). One can double Wittwer's constant to get an estimate for $\{c_J^{(0)}(w)\}$, but as we will see below, it is no longer sharp, though the logarithm remains. The same sequence $\{a_J(w)\}$, but for $w \in A_\infty$, was studied by Vasyunin in [17]. He found the upper and lower Bellman functions for the problem, and constructed explicit optimizing sequences.

Motivated by these results, we consider general sequences $\{c_J^\Phi(w)\}$, defined for $w \in A_2^d$ and a non-negative function Φ on $[1, \infty)$ by

$$(1.4) \quad c_J^\Phi(w) = |J| \Phi(\langle w \rangle_J \langle w^{-1} \rangle_J) \left[\frac{(\Delta_J w)^2}{\langle w \rangle_J^2} + \frac{(\Delta_J w^{-1})^2}{\langle w^{-1} \rangle_J^2} \right].$$

Our ultimate goal is to obtain sharp estimates on $\|\{c_J^\Phi(w)\}\|_C$ in terms of $[w]_{A_2^d}$, i.e. find the largest function k_Φ and the smallest function K_Φ in the following inequality:

$$k_\Phi([w]_{A_2^d}) \leq \sup_{I \in \mathcal{D}} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) \leq K_\Phi([w]_{A_2^d}).$$

Let us keep the names k_Φ and K_Φ for these best functions: for all $Q \geq 1$, we define

$$(1.5) \quad K_\Phi(Q) = \sup_{w: [w]_{A_2^d} = Q} \|\{c_J^\Phi(w)\}\|_C, \quad k_\Phi(Q) = \inf_{w: [w]_{A_2^d} = Q} \|\{c_J^\Phi(w)\}\|_C.$$

In the special case $\Phi(t) = t^\alpha$, we will use the notation K_α and k_α , respectively. Under some fairly mild conditions on Φ , we derive explicit formulas for K_Φ and k_Φ , except for one class of Φ where we simply find good upper estimates. In particular, we compute k_α for all $\alpha \in (-\infty, \infty)$ and K_α for all $\alpha \in (-\infty, \infty) \setminus (1, \frac{3}{2})$. (This seemingly peculiar exclusion is explained in the next section; for $\alpha \in (1, \frac{3}{2})$ we provide an estimate on K_α .)

We deduce our formulas for k_Φ and K_Φ from explicit expressions, or, for some Φ , explicit estimates, for the upper and lower *Bellman functions* for the dyadic sum

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}} c_J^\Phi(w).$$

These functions, formally defined in the next section, are simply the supremum and infimum of this sum, taken over all $w \in A_2^d(I)$ with certain parameters fixed. Once in hand, a Bellman function yields not only a continuum of sharp constants (one for each choice of the parameters), but also the optimizing sequences of weights that realize those constants in the limit.

We continue this discussion in the next section, after key objects have been introduced and the main results stated.

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2. MAIN RESULTS, COROLLARIES, AND DISCUSSION

Let us set notation for the rest of the paper. Alongside Φ , it will be convenient to use the following two functions, also defined on $[1, \infty)$:

$$(2.1) \quad f(s) = \Phi(s^2), \quad h(s) = \frac{f(s)}{s^2}.$$

In addition, if the parameter $Q \geq 1$ is fixed, w is a weight, and $J \in \mathcal{D}$, we let

$$(2.2) \quad L = \sqrt{Q}, \quad s_J(w) = \sqrt{\langle w \rangle_J \langle w^{-1} \rangle_J}, \quad R_J(w) = \frac{(\Delta_J w)^2}{\langle w \rangle_J^2} + \frac{(\Delta_J w^{-1})^2}{\langle w^{-1} \rangle_J^2}.$$

In this notation (1.4) becomes

$$c_J^\Phi(w) = |J| f(s_J(w)) R_J(w).$$

Our headline results are the following two theorems.

Theorem 2.1.

(1) If Φ is increasing and h is convex, then

$$K_\Phi(Q) = 16\Phi(Q) \left(1 - \frac{1}{\sqrt{Q}}\right) + 8 \int_1^Q \frac{\Phi(y)}{y} \left(1 - \frac{1}{\sqrt{y}}\right) dy.$$

(2) If Φ is increasing and h is concave, then

$$K_\Phi(Q) \leq 8(\sqrt{Q} - 1)(3\sqrt{Q} - 1) h(s_0(Q)),$$

where

$$s_0(Q) = \frac{8Q - \sqrt{Q} - 1}{3(3\sqrt{Q} - 1)}.$$

(3) If Φ is decreasing, then

$$K_\Phi(Q) = 8 \int_1^Q \frac{\Phi(y)}{y} dy.$$

Remark 2.2. As the formulation above suggests, we do not know the actual function K_Φ in the case when h is concave. That is because the solution of the corresponding extremal problem differs in structure from that for other Φ ; in particular, it does not arise as the solution of a differential equation. Instead, we provide an upper estimate. As explained in Section 6, the estimate given above can be slightly improved, but the result is far less transparent and likely still not sharp. The estimate in Part (2) is the best known and easy to use, especially when Φ is specified to be a power function (see below).

Our second theorem concerns lower estimates.

Theorem 2.3.

(1) *If h is increasing, then*

$$k_\Phi(Q) = 8 \int_1^Q \frac{\Phi(y)}{y} dy.$$

(2) *If h is decreasing, then*

$$k_\Phi(Q) = 8\Phi(Q) \left(1 - \frac{1}{Q}\right).$$

Note that we make no explicit assumptions of pointwise differentiability, or even continuity, on the part of Φ or h . Let us restate these results for the case $\Phi(t) = t^\alpha$.

Corollary 2.4.

$$K_\alpha(Q) = \begin{cases} \frac{8(2\alpha+1)}{\alpha} Q^\alpha - \frac{32\alpha}{2\alpha-1} Q^{\alpha-1/2} + \frac{8}{\alpha(2\alpha-1)}, & \alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \cup [\frac{3}{2}, \infty); \\ 32Q^{1/2} - 8 \log Q - 32, & \alpha = 1/2; \\ 8 \log Q, & \alpha = 0; \\ \frac{8}{\alpha}(Q^\alpha - 1), & \alpha < 0; \end{cases}$$

$$\begin{aligned} K_\alpha(Q) &\leq 8 \cdot 3^{2-2\alpha} (\sqrt{Q} - 1)(8Q - \sqrt{Q} - 1)^{2\alpha-2} (3\sqrt{Q} - 1)^{3-2\alpha} \\ &= [K_1(Q)]^{3-2\alpha} [K_{3/2}(Q)]^{2\alpha-2}, \end{aligned} \quad \alpha \in (1, \frac{3}{2});$$

$$k_\alpha(Q) = \begin{cases} \frac{8}{\alpha}(Q^\alpha - 1), & \alpha \geq 1; \\ 8(Q^\alpha - Q^{\alpha-1}), & \alpha < 1. \end{cases}$$

Remark 2.5. Thus, for $\alpha > 0, \alpha \notin (1, 3/2)$, the sharp order of growth is $K_\alpha(Q) \approx 8(2+1/\alpha)Q^\alpha$ for large Q . On the other hand, for $\alpha \in (1, 3/2)$ using the formula from Part (2) of Theorem 2.1 amounts simply to log-linear interpolation between the sharp results for $\alpha = 1$ and $\alpha = 3/2$, or, equivalently, to an application of Hölder's inequality. The logarithmic terms in the cases $\alpha = 0$ and $\alpha = 1/2$ correspond to the blow-up of the Nazarov–Volberg estimate (1.2) for these values of α . Lastly, note that we get the same constant for $\|c_J^{(0)}(w)\|_C$ as Wittwer did for $\|a_J(w)\|_C$ in (1.3), even though $c_J^{(0)}(w) = a_J(w) + a_J(w^{-1})$. This reflects the fact that the quantities $a_J(w)$ and $a_J(w^{-1})$ cannot be too large at the same time.

Let us record several related results, all of which are proved in the next section. First, the lower estimates of Theorem 2.3 allow us to obtain a range of equivalent definitions of A_2^d in terms of sequences $\{c_J^\Phi\}$ for all increasing Φ such that $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$. We note that

this limit condition, as well as the inverted lower estimate in Part (ii) have direct analogs in a recent paper [5] concerning equivalent definitions of BMO.

Theorem 2.6. *Assume that Φ is increasing.*

(i) *If $w \in A_2^d$, then the sequence $\{c_J^\Phi(w)\}$ is Carleson and*

$$\|\{c_J^\Phi(w)\}\|_C \leq 8\Phi([w]_{A_2^d}) \log([w]_{A_2^d}).$$

(ii) *If $\{c_J^\Phi(w)\}$ is Carleson and $\lim_{t \rightarrow \infty} \Phi(t) = \infty$, then $w \in A_2^d$ and*

$$[w]_{A_2^d} \leq v(\|\{c_J^\Phi(w)\}\|_C),$$

where $v : [0, \infty) \rightarrow [1, \infty)$ is the inverse to the function $u(Q) = \frac{8}{Q} \int_1^Q \Phi(t) dt$, $Q \geq 1$.

(iii) *If $\lim_{t \rightarrow \infty} \Phi(t) \neq \infty$, then there exists a weight $w \notin A_2^d$ such that $\{c_J^\Phi(w)\}$ is Carleson.*

The main thrust of this theorem is that the condition $\lim_{t \rightarrow \infty} \Phi(t) = \infty$ is necessary and sufficient for the implication “ $\{c_J^\Phi(w)\}$ is Carleson” $\Rightarrow w \in A_2^d$. The quantitative estimates are stated so that they work for all increasing Φ ; as such, they are sharp only on the class of all such Φ . If one has a specific Φ that falls under one of the cases in both Theorem 2.1 and Theorem 2.3, one can, in general, do better by using the estimates from those theorems. In particular, we have the following corollary.

Corollary 2.7.

(i) *If $\alpha > 0$ and $\{c_J^{(\alpha)}(w)\}$ is a Carleson sequence and, then $w \in A_2^d$ and*

$$[w]_{A_2^d} \leq k_\alpha^{-1}(\|\{c_J^{(\alpha)}(w)\}\|_C),$$

where $k_\alpha^{-1} : [0, \infty) \rightarrow [1, \infty)$ is the inverse to k_α given in Theorem 2.13. This estimate is sharp.

(ii) *If $\alpha \leq 0$, there exists a weight $w \notin A_2^d$ such that $\{c_J^{(\alpha)}(w)\}$ is a Carleson sequence.*

The reader will notice that the sequence $\{c_J^\Phi(w)\}$ does not change if we replace w with τw for $\tau > 0$. This zero-degree homogeneity is of central importance in our sharp proofs, but if we are willing to slightly compromise on sharpness, we can easily extend our main theorems to settings with different homogeneity. To illustrate this point, let us examine the sequence

$$c_J^{(\alpha, \beta)}(w) = |J| \langle w \rangle_J^\alpha \langle w^{-1} \rangle_J^\beta R_J(w)$$

for $\alpha \neq \beta$. For this sequence, we can obtain a range of inequalities of the flavor studied in [2].

Corollary 2.8. *For any $w \in A_2^d$ and any α, β such that $\alpha > \beta$,*

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{(\alpha, \beta)}(w) \leq e K_\alpha([w]_{A_2^d}) \langle w^{\alpha-\beta} \rangle_I.$$

Remark 2.9. Of course, the same conclusion holds if we interchange w and w^{-1} . In addition, if $\alpha - \beta \leq 1$, we can replace $\langle w^{\alpha-\beta} \rangle_I$ with $\langle w \rangle_I^{\alpha-\beta}$ by Hölder's inequality.

We now return to Theorems 2.1 and 2.3. To explain where they come from, we need to define the Bellman functions for our problem. Fix $Q \geq 1$ and let

$$\Omega_Q = \{(x_1, x_2) : 1 \leq x_1 x_2 \leq Q\}.$$

For each $x = (x_1, x_2) \in \Omega_Q$ and each $I \in \mathcal{D}$, let

$$E_{Q,x,I} = \{w : w \in A_2^{d,Q}(I), \langle w \rangle_I = x_1, \langle w^{-1} \rangle_I = x_2\}.$$

The upper and lower Bellman functions are defined, respectively, by

$$(2.3) \quad \mathbf{B}_{Q,\Phi}(x_1, x_2) = \sup \left\{ \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) : w \in E_{Q,x,I} \right\}$$

and

$$(2.4) \quad \mathbf{b}_{Q,\Phi}(x_1, x_2) = \inf \left\{ \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) : w \in E_{Q,x,I} \right\}.$$

If Φ is a power function, $\Phi(t) = t^\alpha$, we will write $\mathbf{B}_{Q,\alpha}$ and $\mathbf{b}_{Q,\alpha}$ for $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$, respectively.

One immediately observes that the functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$ are independent of the interval I that formally enters into their definitions. In addition, for any $w \in A_2^{d,Q}$ and any $I \in \mathcal{D}$, we have $(\langle w \rangle_I, \langle w^{-1} \rangle_I) \in \Omega_Q$, thus, the functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$ are defined, at least formally, on Ω_Q . The fact that $E_{Q,x,I}$ is nonempty for every $Q \geq 1$, every interval $I \in \mathcal{D}$, and every $x \in \Omega_Q$ is subsumed in the statements of Theorems 2.10 and 2.11 below.

For a number $L \geq 1$ and a non-negative function f on $[1, \infty)$ formally define:

$$(2.5) \quad A_{L,f}(s) = 16 \left[\frac{f(L)}{L} + \int_1^L \frac{f(z)}{z^2} dz \right] (s-1) - 16 \int_1^s \frac{f(z)}{z^2} (s-z) dz,$$

$$(2.6) \quad a_f(s) = 16 \int_1^s \frac{f(z)}{z} dz.$$

For reasons that will soon be made clear, we will often refer to these two functions as *Bellman candidates*. We are now in a position to state the main theorems about the functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$, which are used in the next section to prove Theorems 2.1 and 2.3.

Theorem 2.10. *In the notation (2.1) and (2.2),*

(1) *If Φ is increasing and h is convex, then*

$$\mathbf{B}_{Q,\Phi}(x_1, x_2) = A_{L,f}(\sqrt{x_1 x_2}).$$

(2) *If Φ is increasing and h is concave, then*

$$\mathbf{B}_{Q,\Phi}(x_1, x_2) \leq 8(\sqrt{x_1 x_2} - 1)(4L - \sqrt{x_1 x_2} - 1) h(s_0(\sqrt{x_1 x_2})),$$

where

$$s_0(s) = \frac{9L^2 - s^2 - s - 1}{3(4L - s - 1)}.$$

(3) *If Φ is decreasing, then*

$$\mathbf{B}_{Q,\Phi}(x_1, x_2) = a_f(\sqrt{x_1 x_2}).$$

Theorem 2.11. *In the notation (2.1) and (2.2),*

(1) *If h is increasing, then*

$$\mathbf{b}_{Q,\Phi}(x_1, x_2) = a_f(\sqrt{x_1 x_2}).$$

(2) *If h is decreasing, then*

$$\mathbf{b}_{Q,\Phi}(x_1, Q/x_1) = 8\Phi(Q) \left(1 - \frac{1}{Q}\right).$$

Remark 2.12. Note that in all cases in these theorems, all integrals involved in the definitions of $A_{L,f}$ and a_f are well-defined.

To summarize: in Parts (1) and (3) of Theorem 2.10 and in Part (1) of Theorem 2.11, we find the exact Bellman functions at every point of Ω_Q ; in Part (2) of Theorem 2.10, we provide a point-wise majorant for $B_{Q,\Phi}$; and in Part (2) of Theorem 2.11, we find $b_{Q,\Phi}$ only on the boundary curve $x_1x_2 = Q$ (which, however, is enough to compute k_Φ for this case).

For $\Phi(t) = t^\alpha$, these formulas become

Corollary 2.13. *Let $s = \sqrt{x_1x_2}$.*

$$B_{Q,\alpha}(x_1, x_2) = \begin{cases} \frac{8}{\alpha(2\alpha-1)}(1-s^{2\alpha}) + \frac{32\alpha}{2\alpha-1}L^{2\alpha-1}(s-1), & \alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \cup [\frac{3}{2}, \infty); \\ 32(s-1) - 16 \log L + 16s \log\left(\frac{L}{s}\right), & \alpha = 1/2; \\ 16 \log s, & \alpha = 0; \\ \frac{8}{\alpha}(s^{2\alpha} - 1), & \alpha < 0; \end{cases}$$

$$B_{Q,\alpha}(x_1, x_2) \leq [B_{Q,1}(x_1, x_2)]^{3-2\alpha} [B_{Q,3/2}(x_1, x_2)]^{2\alpha-2}, \quad \alpha \in (1, \frac{3}{2});$$

$$b_{Q,\alpha}(x_1, x_2) = \frac{8}{\alpha}(s^{2\alpha} - 1), \quad \alpha \geq 1;$$

$$b_{Q,\alpha}(x_1, Q/x_1) = 8L^{2\alpha-2}(L^2 - 1), \quad \alpha < 1.$$

Before proceeding with the proofs of these theorems and corollaries, we briefly discuss the principal method at play and make several related remarks.

2.1. The method. The Bellman function technique in analysis consists of finding special functions with designated size and convexity properties, to aid in induction-based proofs. In martingale settings, such as ours, the first Bellman-function proofs can be found in the work of Burkholder [4]. Later, the technique was applied, in a significantly different form and under the current name, to many problems in harmonic analysis by Nazarov, Treil, and Volberg [11, 10, 12], followed by others; the reader can consult the lecture notes [18] for details. We also note the work of Osękowski on sharp inequalities for martingales [14, 15], which implements what can be termed a mixed Burkholder–Bellman method.

An important distinction exists between two kinds of Bellman functions. One kind is the *true* Bellman functions, which are defined as solutions of extremal problems such as (2.3) and (2.4). They provide complete information about the inequality in question, including the structure of optimizers, but may be difficult to compute. The other and much more prevalent kind are *Bellman-type* functions, which are useful substitutes with properties similar to those of the true function. These are non-unique and much easier to find; however, they rarely produce completely sharp estimates and never the exact optimizers. Inequality (1.2), due to Beznosova and Nazarov–Treil, is a good example of the use of Bellman-type functions. Observe that the bound obtained is sharp in $[w]_{A_2^d}$, but sub-optimal in the numerical factor.

Recent advances in finding true Bellman functions have made them viable as stand-alone objects of study, ahead of the inequalities they imply; that perspective is maintained here. To find the true function, one often needs to solve a PDE (a notable exception is the work of Melas and co-authors [6, 7, 8], where one determines the function starting from external information about the optimizers). The functions $A_{L,f}$ and a_f from (2.5) and (2.6) are two homogenized solutions of such a PDE. Overall, the Bellman-related computations in this paper

are somewhat similar to those in Vasyunin's proof of Buckley's inequality [17], though they are much more general. We also incorporate elements from [16] and [19], particularly in the construction of optimizing sequences.

The reader familiar with the method will find in the paper at least two technical novelties that are likely to be useful in other settings. First, in Section 5, when verifying key inequalities for our Bellman candidates (the so-called "main inequalities"), we prove statements that are both stronger and simpler than those required. That may seem like a dangerous overreach, as true-Bellman proofs are typically very tight. However, the inequalities we need and the inequalities we actually prove are extremized by the same configurations of the variables involved, and the slack we introduce happens away from those configurations. Second, in Section 7, when constructing the optimizer for one of the candidates, we only do so for a special selection of points in the domain Ω_Q , and then use the *a priori* continuity of our Bellman functions to get the desired result. Doing so saves us the messy work of constructing the optimizer for every point of the domain.

2.2. Extensions. The results presented above suggest several possible generalizations. The first such generalization concerns computing the functions k_Φ and K_Φ for an arbitrary Φ . The main obstacle, of course, is that we do not yet know K_Φ when $h(s) = \Phi(s^2)/s^2$ is concave. Section 6 discusses a possible approach to computing the Bellman function $B_{Q,\Phi}$ (and, hence, K_Φ) for such Φ . It is plausible that once that solution is in hand, and one thus has three different Bellman candidates corresponding to the three parts of Theorem 2.10, one can deal with an arbitrary Φ by gluing those candidates in an appropriate manner at the points where Φ and/or h change behavior.

One can ask whether our results have an analog in higher dimensions. Of course, the definition of $\{c_J^\Phi(w)\}$ has to change: we need to replace expressions such as $\langle w \rangle_{J^-} - \langle w \rangle_{J^+}$, where J a dyadic interval, with $\langle w \rangle_{J^-} - \langle w \rangle_{P(J)}$, where J is a dyadic cube and $P(J)$ its unique parent. Beyond that, one must adapt the inductive argument at the center of the proofs of Theorems 2.10 and 2.11 to the situation where one must keep track of 2^n points from the domain Ω_Q on every step of the induction, as opposed to just two points before. A somewhat similar challenge, but without regard for sharpness, was successfully handled in [5].

Lastly, one may be interested in studying the sequence $\{c_J^\Phi(w)\}$ when w is an A_p weight with $p \neq 2$ or a reverse Hölder weight. It seems likely that when the definition of c_J^Φ is properly symmetrized, most of our analysis will go through in such settings, though we have not explored that.

2.3. Linearity of sharp estimates. Let us turn things around a bit and consider the function K_Φ as an operator on Φ , taking in and returning non-negative functions on $[1, \infty)$. For $t \geq 1$, let

$$(T\Phi)(t) = K_\Phi(t).$$

Directly from its definition, T is sublinear on the cone of all non-negative Φ , meaning that for any $a_1, a_2 \geq 0$, $T(a_1\Phi_1 + a_2\Phi_2) \leq a_1T\Phi_1 + a_2T\Phi_2$. However, Theorem 2.1 shows that this operator in fact behaves linearly on broad classes of Φ . For example, for all increasing Φ such that $h(s) = \Phi(s^2)/s^2$ is concave, $T\Phi$ is given by the linear (in Φ) expression in Part (1) of Theorem 2.1. That means that if Φ_1, Φ_2 are two such functions, then $T(\Phi_1 + \Phi_2) = T\Phi_1 + T\Phi_2$, which in terms of the original formulation means that to get a sharp estimate for the Carleson norm of $\{c_J^{(\Phi_1 + \Phi_2)}\}$ we need simply add the sharp estimates for Φ_1 and Φ_2 . In particular, if P is a polynomial with non-negative coefficients a_0, \dots, a_n , then by Corollary 2.4, $K_P(Q) = \sum_{j=0}^n a_j K_j(Q)$.

Similar observations can be made about the function k_Φ . What accounts for this linearity phenomenon is the nature of optimizers, i.e., those sequences of weights on which the supremum and infimum are attained (in the limit) in the definitions (2.3) and (2.4) of the functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$. As shown in Section 7 below, these optimizers do not depend on the exact Φ involved, but only on the differential structure of the candidates $A_{L,f}$ and a_f .

2.4. Outline. The rest of the paper is organized as follows: in Section 3, we prove all results stated in this section, except Theorem 2.10 and Theorem 2.11(1); in Section 4, we establish key properties of the Bellman functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$ and derive the Bellman candidates $A_{L,f}$ and a_f ; in Section 5, we verify that $A_{L,f}$ and a_f bound $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$ from above or below, as appropriate; Section 6 contains the proof of Theorem 2.10(2), as well as a brief discussion of that case; finally, in Section 7, we present a detailed construction of optimizers for $A_{L,f}$ and a_f , thus finishing the proofs of Theorems 2.10 and 2.11.

3. SHORTER PROOFS

The order of proofs is as follows: Theorem 2.11(2), Theorem 2.1, Theorem 2.3, Theorem 2.6, Corollary 2.7, and Corollary 2.8.

Proof of Theorem 2.11, Part (2). Take any $Q \geq 1$, any interval $I \in \mathcal{D}$, and any $w \in A_2^{d,Q}$ such that $s_I^2(w) = \langle w \rangle_I \langle w^{-1} \rangle_I = Q$. Since the function $h(s) = f(s)/s^2$ is decreasing, we have

$$(3.1) \quad \begin{aligned} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) &= \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |J| f(s_J(w)) R_J(w) \\ &\geq \frac{f(L)}{L^2} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |J| s_J^2(w) R_J(w) \geq \frac{f(L)}{L^2} \mathbf{b}_{Q,1}(\langle w \rangle_I, \langle w^{-1} \rangle_I), \end{aligned}$$

where $\mathbf{b}_{Q,1}$ is the Bellman function defined by (2.4) with $\Phi(t) = t$ or, equivalently, $f(s) = s^2$. For this f , $h(s) = 1$, and we can apply the first part of the theorem: $\mathbf{b}_{Q,1}(x_1, x_2) = 8(x_1 x_2 - 1)$. Therefore, returning to the original Φ and f , we obtain

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) \geq 8 \frac{f(L)}{L^2} (L^2 - 1) = 8 \Phi(Q) \left(1 - \frac{1}{Q}\right),$$

which means that $\mathbf{b}_{Q,\Phi}(x_1, Q/x_1) \geq 8 \Phi(Q) (1 - \frac{1}{Q})$.

To prove equality, take any I and any $x_1 > 0$ and consider the weight

$$(3.2) \quad w^*(t) = \begin{cases} x_1 \left(1 - \sqrt{1 - \frac{1}{Q}}\right), & t \in I^-, \\ x_1 \left(1 + \sqrt{1 - \frac{1}{Q}}\right), & t \in I^+. \end{cases}$$

We compute: $\langle w^* \rangle_I = x_1$, $\langle (w^*)^{-1} \rangle_I = Q/x_1$, and $R_I(w^*) = 8(1 - 1/Q)$; also note that w^* is constant on each $J \in \mathcal{D}(I^-) \cup \mathcal{D}(I^+)$. Therefore, $[w^*]_{A_2^{d,Q}(I)} = Q$ and

$$\mathbf{b}_{Q,\Phi}(x_1, Q/x_1) \leq \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w^*) = \frac{1}{|I|} |I| f(s_I(w^*)) R_I(w^*) = 8 \Phi(Q) \left(1 - \frac{1}{Q}\right).$$

□

Proof of Theorem 2.1. We will derive this theorem as a corollary of Theorem 2.10. Take $Q \geq 1$ and a $w \in A_2^d$ such that $Q = [w]_{A_2^d}$. For any $I \in \mathcal{D}$, by the definition of $\mathbf{B}_{Q,\Phi}$,

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) \leq \mathbf{B}_{Q,\Phi}(\langle w \rangle_I, \langle w^{-1} \rangle_I) \leq \sup_{x \in \Omega_Q} \mathbf{B}_{Q,\Phi}(x_1, x_2).$$

Therefore, $\|\{c_J^\Phi(w)\}\|_C \leq \sup_{x \in \Omega_Q} \mathbf{B}_{Q,\Phi}(x_1, x_2)$ and, finally,

$$K_\Phi(Q) = \sup_{[w]_{A_2^d} = Q} \|\{c_J^\Phi(w)\}\|_C \leq \sup_{x \in \Omega_Q} \mathbf{B}_{Q,\Phi}(x_1, x_2).$$

The right-hand side expressions in all three statements of Theorem 2.10 attain their maxima when $s = \sqrt{Q} = L$. For Parts (1) and (3), this is because both $A_{L,f}$ and, respectively, a_f are increasing functions. For Part (2), we have

$$8(s-1)(4L-s-1)h(s_0(s)) = \frac{8(s-1)(4L-s-1)^3}{9(9L^2-s^2-s-1)^2} f(s_0(s)).$$

It is easy to verify that the fraction in front of f is increasing in s , as is the function s_0 ; on the other hand, f is increasing by assumption.

Therefore, we obtain the three statements of Theorem 2.1 — but with “ \leq ” instead of “ $=$ ” in Parts (1) and (3) — by setting $s = \sqrt{Q}$ in the three respective statements of Theorem 2.10 and changing the variable in the integrals.

To prove that the first and third statements hold with equality, note that by the definition of $\mathbf{B}_{Q,\Phi}$, for any $Q \geq 1$, any $x_1 > 0$ and any interval I , there exists a sequence of $A_2^d(I)$ -weights $\{w_n\}$, such that for each n $[w_n]_{A_2^d(I)} = Q$, $\langle w_n \rangle_I = x_1$, $\langle w_n \rangle_I \langle w_n^{-1} \rangle_I = Q$, and

$$\lim_{n \rightarrow \infty} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n) = \mathbf{B}_{Q,\Phi}(x_1, Q/x_1).$$

Let us extend each w_n to all of \mathbb{R} periodically by replicating it on each dyadic interval of length $|I|$; keep the name w_n for the extension. Clearly, $[w_n]_{A_2^d} = [w_n]_{A_2^d(I)}$, which means that the left-hand side does not exceed $K_\Phi(Q)$. On the other hand, by Theorem 2.10 the right-hand side equals $\mathcal{A}_{L,f}(\sqrt{Q})$ in Part (1) and $a_f(\sqrt{Q})$ in Part (3). The proof is complete. \square

Proof Theorem 2.3. We will assume the statements of Theorem 2.11, Lemma 7.1, and Remark 7.2. The last two come from Section 7, which itself is self-contained.

Proof of Part (1). Take any $Q \geq 1$ and any $w \in A_2^d$ such that $[w]_{A_2^d} = Q$. There exists a sequence of intervals $\{I_n\}$ such that $\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n} \rightarrow Q$ as $n \rightarrow \infty$. By the definition of $\mathbf{b}_{Q,\Phi}$,

$$\frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} c_J^\Phi(w) \geq \mathbf{b}_{Q,\Phi}(\langle w \rangle_{I_n}, \langle w^{-1} \rangle_{I_n}) = a_f\left(\sqrt{\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n}}\right).$$

Therefore,

$$\|\{c_J^\Phi\}\|_C \geq \lim_{n \rightarrow \infty} a_f\left(\sqrt{\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n}}\right) = a_f(\sqrt{Q}),$$

since a_f is continuous on $[1, \infty)$. Taking the infimum over all w with $[w]_{A_2^d} = Q$ gives

$$k_\Phi(Q) \geq a_f(\sqrt{Q}).$$

To prove the converse inequality, we use the sequence $\{w_n^{(L,L)}\}$ of weights on $(0, 1)$ constructed in Section 7. That sequence is given by (7.6) and (7.7) with $s_0 = L$; let us call it simply $\{w_n\}$

here. Let $I = (0, 1)$. Lemma 7.1 asserts that $[w_n]_{A_2^d(I)} = Q$ for all n and that

$$\lim_{n \rightarrow \infty} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n) = a_f(\sqrt{Q}).$$

In addition, from Remark 7.2,

$$\sup_{R \in \mathcal{D}(I)} \frac{1}{|R|} \sum_{J \in \mathcal{D}(R)} c_J^\Phi(w_n) \leq \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n).$$

Extend each w_n periodically to all of \mathbb{R} . Clearly, for each n , $[w_n]_{A_2^d} = Q$ and

$$\|\{c_J^\Phi(w_n)\}\|_C = \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w_n).$$

The left-hand side is no less than $k_\Phi(Q)$, while the right-hand side converges to $a_f(\sqrt{Q})$. After changing the variable in the integral $\int_1^L f(z)/z \, dz$, we obtain the statement in Part (1) of the theorem.

Proof of Part (2). The proof proceeds along the lines of that of Part (1), but also uses key ingredients from the proof of Theorem 2.11(2). First, take $Q \geq 1$ and any $w \in A_2^d$ such that $[w]_{A_2^d} = Q$. There exists a sequence of dyadic intervals $\{I_n\}$ such that $\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n} \rightarrow Q$. Now, as in (3.1),

$$\begin{aligned} \|\{c_J^\Phi(w)\}\|_C &\geq \frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} c_J^\Phi(w) = \frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} |J| f(s_J(w)) R_J(w) \\ &\geq \frac{f(L)}{L^2} \frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} |J| s_J^2(w) R_J(w) \geq \frac{f(L)}{L^2} \mathbf{b}_{Q,1}(\langle w \rangle_{I_n}, \langle w^{-1} \rangle_{I_n}) \\ &= 8 \frac{f(L)}{L^2} \left(\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n} - 1 \right) \xrightarrow{n \rightarrow \infty} 8\Phi(Q) \left(1 - \frac{1}{Q} \right). \end{aligned}$$

Therefore, $k_\Phi(Q) \geq 8\Phi(Q)(1 - \frac{1}{Q})$. To establish the converse, take an interval I and let w^* be the weight on I defined by (3.2). Extend w^* to all of \mathbb{R} as in Part (1). Then $[w^*]_{A_2^d} = Q$ and

$$\|\{c_J^\Phi(w)\}\|_C = \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w^*) = 8\Phi(Q) \left(1 - \frac{1}{Q} \right).$$

The left-hand side is no smaller than $k_\Phi(Q)$, hence the proof is complete. \square

Proof of Theorem 2.6. We first obtain estimates on K_Φ and k_Φ that work for all increasing Φ . Assume that $w \in A_2^d$ and let $[w]_{A_2^d} = Q$. For all $s \in [1, L]$, $f(s) \frac{s^2}{L^2} \leq f(s) \leq f(L)$. Therefore, for $c_J^\Phi(w) = |J| f(s_J(w)) R_J(w)$ we get

$$c_J^{\Phi_-}(w) \leq c_J^\Phi(w) \leq c_J^{\Phi_+}(w),$$

where we have set $\Phi_-(t) = \Phi(t) \frac{t}{Q}$ and $\Phi_+(t) = \Phi(Q)$. Hence,

$$k_{\Phi_-}(Q) \leq k_\Phi(Q), \quad K_\Phi(Q) \leq K_{\Phi_+}(Q).$$

The function $\Phi_-(t)/t = \Phi(t)/Q$ is increasing, so by Theorem 2.3(1)

$$k_{\Phi_-}(Q) = 8 \int_1^Q \frac{\Phi_-(t)}{t} \, dt = \frac{8}{Q} \int_1^Q \Phi(t) \, dt.$$

On the other hand, Φ_+ is decreasing, so by Theorem 2.1(3)

$$K_{\Phi_+}(Q) = 8 \int_1^Q \frac{\Phi_+(t)}{t} dt = 8\Phi(Q) \log Q.$$

This already proves part (i). To prove (ii), we observe that if $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$ (write $\Phi(\infty) = \infty$), then the function k_{Φ_-} defined above is strictly increasing and $k_{\Phi_-}(\infty) = \infty$. Therefore, k_{Φ_-} is invertible on $[1, \infty)$.

Now, assume that $\{c_J^\Phi(w)\}$ is Carleson, but $w \notin A_2^d$. Take $I \in \mathcal{D}$ and let $w^I = w\chi_I$. For $n \geq 0$, let w_n^I be the truncation of w^I at the n -th dyadic generation:

$$w_n^I = \sum_{J \in \mathcal{D}_n(I)} \langle w \rangle_J \chi_J.$$

Extend w_n^I periodically to all of \mathbb{R} ; still call it w_n^I . Observe that $w_n^I \in A_2^d$ and $\|c_J^\Phi(w_n^I)\|_C \leq \|c_J^\Phi(w)\|_C$. Therefore,

$$(3.3) \quad k_{\Phi_-}([w_n^I]_{A_2^d}) \leq \|c_J^\Phi(w)\|_C.$$

Since $w \notin A_2^d$, $\sup_{I \in \mathcal{D}} (\lim_{n \rightarrow \infty} [w_n^I]_{A_2^d}) = \infty$. Since $k_{\Phi_-}(\infty) = \infty$, if we take first the limit in (3.3) as $n \rightarrow \infty$ and then, if still necessary, the supremum over all I , we obtain a contradiction. Therefore, $w \in A_2^d$; furthermore, $k_{\Phi_-}([w]_{A_2^d}) \leq \|\{c_J^\Phi(w)\}\|_C$. Inverting k_{Φ_-} (called u in the statement of the theorem) completes the proof of part (ii).

To prove (iii), first observe that if $\Phi(\infty) \neq \infty$, then

$$\|\{c_J^\Phi(w)\}\|_C \leq \Phi(\infty) \|\{|J|R_J(w)\}\|_C.$$

We will now present an explicit weight $w \notin A_2^d$ such that the sequence $\{|J|R_J(w)\}$ is Carleson. For $t \in (0, 1)$, let

$$w(t) = \sum_{k=1}^{\infty} 2^k k^{-2} \cdot \chi_{(2^{-k}, 2^{-k+1})}(t).$$

Now, extend w periodically to \mathbb{R} . For $n \geq 0$, let $I_n = (0, 2^{-n})$. We compute

$$\langle w \rangle_{I_n} = 2^n \sum_{k=n+1}^{\infty} k^{-2} \geq 2^n n^{-1}, \quad \langle w^{-1} \rangle_{I_n} = 2^n \sum_{k=n+1}^{\infty} k^2 4^{-k} \geq \frac{1}{3} n^2 2^{-n}.$$

Therefore, $\langle w \rangle_{I_n} \langle w^{-1} \rangle_{I_n} \geq n/3$, and so $w \notin A_2^d$.

Now, clearly

$$\|\{|J|R_J(w)\}\|_C = \sup_{I \in \mathcal{D}(0,1)} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |J|R_J(w).$$

Observe, that the only intervals in $\mathcal{D}(0, 1)$ on which w is not constant — and, thus, on which $\sum_{J \in \mathcal{D}(I)} |J|R_J(w) \neq 0$ — are the intervals I_n for $n \geq 0$. For each n we have

$$\frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} |J|R_J(w) = \frac{1}{|I_n|} \sum_{k=n}^{\infty} |I_k|R_{I_k}(w).$$

For any weight w and any interval J , $\langle w \rangle_{J^\pm} \leq 2\langle w \rangle_J$, so $R_J(w) = (\langle w \rangle_{J^-} - \langle w \rangle_{J^+})^2 / \langle w \rangle_J^2 + (\langle w^{-1} \rangle_{J^-} - \langle w^{-1} \rangle_{J^+})^2 / \langle w^{-1} \rangle_J^2 \leq 8$. We conclude that

$$\|\{|J|R_J(w)\}\|_C = \sup_n \frac{1}{|I_n|} \sum_{J \in \mathcal{D}(I_n)} |J|R_J(w) \leq 8 \cdot 2^n \sum_{k=n}^{\infty} 2^{-k} = 16,$$

finishing the proof. \square

Proof of Corollary 2.7. The only part of this corollary not contained in Theorem 2.7 is that for $\Phi(t) = t^\alpha$, Corollary 2.4 gives exact expressions for k_α and we do not need the inequality $k_\Phi(Q) \geq \frac{8}{Q} \int_1^Q \Phi(t) dt$. Observe that k_α is indeed invertible for all $\alpha > 0$. \square

Proof of Corollary 2.8. The statement of the corollary is an immediate consequence of the following lemma. This lemma extends the results of Beznosova [1], who proved it with $\gamma = 1$ and 4 in place of e , and Moraes, who proved it in his thesis (also see [3]) for all $\gamma \geq 0$, but still with the constant 4. The proof given works in any dimension, though we need it only in dimension 1.

Lemma 3.1. *Let w be a weight such that w^{-1} is also a weight. Assume $\{c_J\}_{J \in \mathcal{D}}$ is a Carleson sequence with norm B . Then for any $\gamma \geq 0$ and each $I \in \mathcal{D}$,*

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J \frac{1}{\langle w^{-1} \rangle_J^\gamma} \leq e B \langle w^\gamma \rangle_I.$$

Proof. The case $\gamma = 0$ is obvious, so assume $\gamma > 0$. Observe that for any interval I , any $J \in \mathcal{D}(I)$, and any $p > 1/\gamma$,

$$\langle w^{-1} \rangle_J^{-\gamma} \leq \langle w^{1/p} \rangle_J^{p\gamma} \leq \inf_{x \in J} [M(w^{1/p} \chi_I)]^{p\gamma}(x).$$

Here M is the dyadic maximal function, $M\varphi(x) \stackrel{\text{def}}{=} \sup_{x \in R \in \mathcal{D}} \langle |\varphi| \rangle_R$. Recall that for $p > 1$ the norm of M on $L^p(\mathbb{R}^n)$ equals $p/(p-1)$.

Now, using Carleson Lemma 1.1 with $F(x) = [M(w^{1/p} \chi_I)]^{p\gamma}$ gives

$$\begin{aligned} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J \frac{1}{\langle w^{-1} \rangle_J^\gamma} &\leq \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J \inf_{x \in J} [M(w^{1/p} \chi_I)]^{p\gamma}(x) \leq B \frac{1}{|I|} \int_{\mathbb{R}} [M(w^{1/p} \chi_I)]^{p\gamma}(x) dx \\ &\leq B \left(\frac{p\gamma}{p\gamma-1} \right)^{p\gamma} \frac{1}{|I|} \|w^{1/p} \chi_I\|_{L^{p\gamma}(\mathbb{R})}^{p\gamma} = B \left(\frac{p\gamma}{p\gamma-1} \right)^{p\gamma} \langle w^\gamma \rangle_I. \end{aligned}$$

Taking the limit as $p \rightarrow \infty$, we obtain the statement of the lemma. \square

To prove the corollary, write $\gamma = \alpha - \beta$. Then

$$c_J^{(\alpha, \beta)}(w) = |J| \langle w \rangle_J^\alpha \langle w^{-1} \rangle_J^\beta R_J(w) = |J| (\langle w \rangle_J \langle w^{-1} \rangle_J)^\alpha \langle w^{-1} \rangle_J^\gamma R_J(w) = c_J^{(\alpha)}(w) \langle w^{-1} \rangle_J^{-\gamma}.$$

Since $\{c_J^{(\alpha)}(w)\}$ is a Carleson sequence with norm not exceeding $K_\alpha([w]_{A_2^d})$, the result follows from Lemma 3.1. \square

4. NECESSARY CONDITIONS ON BELLMAN CANDIDATES; INDUCTION ON SCALES

When seeking the Bellman function for an inequality, one typically determines key conditions that this function must satisfy directly from its definition and then derives (or otherwise presents) a *candidate* function that possess these properties. The candidate is then shown to equal the true Bellman function – or at least bound it from above or below, as appropriate. The actual proof of the inequality consists of revealing the exact relationship between the Bellman candidate and the Bellman function and, technically, does not require one to know where the candidate comes from. However, the process of constructing a candidate both illustrates the main inductive component of the proof and is intrinsically tied with the construction of optimizers in the original inequality. Accordingly, we present it in some detail.

The following lemma lists three key properties of $\mathbf{B}_{Q, \Phi}$ and $\mathbf{b}_{Q, \Phi}$ that follow from definitions (2.3) and (2.4).

Lemma 4.1. *The functions $\mathbf{B}_{Q,\Phi}$ and $\mathbf{b}_{Q,\Phi}$ satisfy*

(1) **Main inequality.** *For all points $x^-, x^+ \in \Omega_Q$ such that $(x^- + x^+)/2 \in \Omega_Q$,*

(4.1)

$$\mathbf{B}_{Q,\Phi}\left(\frac{x^- + x^+}{2}\right) \geq \frac{1}{2}\mathbf{B}_{Q,\Phi}(x^-) + \frac{1}{2}\mathbf{B}_{Q,\Phi}(x^+) + \Phi(x_1x_2) \left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} \right],$$

$$(4.2) \quad \mathbf{b}_{Q,\Phi}\left(\frac{x^- + x^+}{2}\right) \leq \frac{1}{2}\mathbf{b}_{Q,\Phi}(x^-) + \frac{1}{2}\mathbf{b}_{Q,\Phi}(x^+) + \Phi(x_1x_2) \left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} \right].$$

(2) **Boundary condition.** *For all $x_1 \in (0, \infty)$,*

$$(4.3) \quad \mathbf{B}_{Q,\Phi}(x_1, 1/x_1) = 0, \quad \mathbf{b}_{Q,\Phi}(x_1, 1/x_1) = 0.$$

(3) **Homogeneity.** *For all $x \in \Omega_Q$,*

$$(4.4) \quad \mathbf{B}_{Q,\Phi}(x_1, x_2) = \mathbf{B}_{Q,\Phi}(\sqrt{x_1x_2}, \sqrt{x_1x_2}), \quad \mathbf{b}_{Q,\Phi}(x_1, x_2) = \mathbf{b}_{Q,\Phi}(\sqrt{x_1x_2}, \sqrt{x_1x_2}).$$

Proof. It suffices to prove these statements for $\mathbf{B}_{Q,\Phi}$ as the proof for $\mathbf{b}_{Q,\Phi}$ is almost identical.

To prove (4.1), first observe that for any point $x = (x_1, x_2) \in \Omega_Q$, there exists a weight $w \in A_2^{d,Q}(I)$ such that $\langle w \rangle_I = x_1$ and $\langle w^{-1} \rangle_I = x_2$. For instance, one can take

$$(4.5) \quad w(t) = \begin{cases} x_1 \left(1 - \sqrt{1 - \frac{1}{x_1x_2}}\right), & t \in I^-, \\ x_1 \left(1 + \sqrt{1 - \frac{1}{x_1x_2}}\right), & t \in I^+. \end{cases}$$

Indeed, being constant on I^\pm , this weight is in $A_2^d(I^\pm)$ with $[w]_{A_2^d(I^\pm)} = 1$. Since $x \in \Omega_Q$ and

$$A_2^{d,Q}(I) = \{w : w|_{I^-} \in A_2^{d,Q}(I^-), w|_{I^+} \in A_2^{d,Q}(I^+), \langle w \rangle_I \langle w^{-1} \rangle_I \leq Q\},$$

the conclusion follows. This means that the supremum and infimum in the definitions (2.3) and (2.4) are taken over a non-empty set.

Now, for any $w \in A_2^{d,Q}(I)$,

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) = \frac{1}{2} \frac{1}{|I^-|} \sum_{J \in \mathcal{D}(I^-)} c_J^\Phi(w) + \frac{1}{2} \frac{1}{|I^+|} \sum_{J \in \mathcal{D}(I^+)} c_J^\Phi(w) + f(s_I(w))R_I(w),$$

where we have used notation (2.2). Inequality (4.1) now follows from considering any weight $w^* \in A_2^{d,Q}(I)$ that realizes (or almost realizes) the supremum in (2.3) on both I^- and I^+ (note that $w^*|_{I^-}$ and $w^*|_{I^+}$ can be chosen independently of each other).

The boundary condition (4.3) holds since the only functions w in $A_2^d(I)$ satisfying $\langle w \rangle_I = 1/\langle w^{-1} \rangle_I$ are constants.

Lastly, to show (4.4), take any $\tau > 0$ and replace w with τw in the definitions (2.3) and (2.4). The class $A_2^{d,Q}$ is invariant under this transformation, as is the sum $\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w)$. Therefore,

$$\mathbf{B}_{Q,\Phi}(\tau x_1, \tau^{-1} x_2) = \mathbf{B}_{Q,\Phi}(x_1, x_2),$$

and setting $\tau = \sqrt{x_2/x_1}$ finishes the proof. \square

The central point in Bellman analysis on martingales is that if one has *any* function B on Ω_Q that satisfies (4.1) and (4.3), that function is automatically a majorant of $\mathbf{B}_{Q,\Phi}$; a similar conclusion applies to $\mathbf{b}_{Q,\Phi}$. Specifically, we have the following lemma, which implements in our setting the main part of any Bellman-function argument, sometimes referred to as “Bellman induction” or “induction on scales.”

Lemma 4.2. *Let B and b be functions on Ω_Q satisfying*

$$(4.6) \quad B\left(\frac{x^- + x^+}{2}\right) \geq \frac{1}{2}B(x^-) + \frac{1}{2}B(x^+) + \Phi(x_1x_2) \left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} \right]$$

and

$$(4.7) \quad b\left(\frac{x^- + x^+}{2}\right) \leq \frac{1}{2}b(x^-) + \frac{1}{2}b(x^+) + \Phi(x_1x_2) \left[\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} \right],$$

respectively, for all points $x^-, x^+ \in \Omega_Q$ such that $(x^- + x^+)/2 \in \Omega_Q$, as well as

$$(4.8) \quad B(x_1, 1/x_1) = 0, \quad b(x_1, 1/x_1) = 0, \quad \forall x_1 \in (0, \infty).$$

Assume that Φ is a non-negative, bounded function on $[1, Q]$. Then

$$B(x) \geq \mathbf{B}_{Q,\Phi}(x), \quad b(x) \leq \mathbf{b}_{Q,\Phi}(x), \quad \forall x \in \Omega_Q.$$

Proof. First, we obtain elementary estimates on B and b . Take any $x \in \Omega_Q$ and write $x = (x^- + x^+)/2$, where, as in (4.5),

$$x_1^\pm = x_1 \left(1 \pm \sqrt{1 - \frac{1}{x_1x_2}}\right), \quad x_2^\pm = x_2 \left(1 \mp \sqrt{1 - \frac{1}{x_1x_2}}\right).$$

We have $x_1^-x_2^- = x_1^+x_2^+ = 1$ and $(x_1^- - x_1^+)^2/x_1^2 + (x_2^- - x_2^+)^2/x_2^2 = 8(1 - 1/(x_1x_2))$. Let $M = \|\Phi\|_{L^\infty([1,Q])}$. Using (4.6) and noting that $B(x^-) = B(x^+) = 0$ by (4.8), we obtain

$$B(x) \geq 8\Phi(x_1x_2) \left(1 - \frac{1}{x_1x_2}\right) \geq 0.$$

Similarly,

$$b(x) \leq 8\Phi(x_1x_2) \left(1 - \frac{1}{x_1x_2}\right) \leq 8M \left(1 - \frac{1}{x_1x_2}\right).$$

Now, again take any $x \in \Omega_Q$ and any interval I . Let $w \in A_2^{d,Q}(I)$ be any weight such that $\langle w \rangle_I = x_1$ and $\langle w^{-1} \rangle_I = x_2$. For any $J \in \mathcal{D}(I)$, let $x_J = (\langle w \rangle_J, \langle w^{-1} \rangle_J)$; since $w \in A_2^{d,Q}(I)$, $x_J \in \Omega_Q$. Applying (4.6) repeatedly, we have

$$\begin{aligned} B(x) &\geq \frac{1}{2}B(x_{I-}) + \frac{1}{2}B(x_{I+}) + \Phi(x_1x_2) R_I(w) \\ &\geq \frac{1}{|I|} \sum_{J \in \mathcal{D}_n(I)} |J| B(x_J) + \sum_{k=0}^{n-1} \sum_{J \in \mathcal{D}_n(I)} f(s_J) R_J(w). \end{aligned}$$

Since $B \geq 0$, we can drop the first sum and then take the limit as $n \rightarrow \infty$. Taking the supremum over all $w \in A_2^{d,Q}(I)$ with $\langle w \rangle_I = x_1$ and $\langle w^{-1} \rangle_I = x_2$ proves the lemma for B .

The argument for b is similar. Using (4.7) repeatedly, we get

$$(4.9) \quad b(x) \leq \frac{1}{|I|} \sum_{J \in \mathcal{D}_n(I)} |J| b(x_J) + \sum_{k=0}^{n-1} \sum_{J \in \mathcal{D}_n(I)} f(s_J) R_J(w).$$

The first sum is bounded by $\frac{8M}{|I|} \sum_{J \in \mathcal{D}_n(I)} |J| \left(1 - \frac{1}{s_J^2}\right)$, which converges to 0 as $n \rightarrow \infty$ by the Lebesgue differentiation theorem and the Lebesgue dominated convergence theorem. Taking the limit as $n \rightarrow \infty$ and then the infimum over all appropriate w completes the proof. \square

Remark 4.3. The reader may wonder if the condition that Φ be bounded is necessary in this lemma. It is clearly not needed for the conclusion about B . For the lower candidate b , however, some technical assumption is needed in order to disregard the first sum in the right-hand side of (4.9). An alternative would be to assume that b is continuous on Ω_Q , as one can then run the induction only for dyadically-simple weights (for which the induction is finite) and then approximate an arbitrary weight by dyadically-simple ones. In any case, this distinction is inconsequential: first, in all cases of Theorem 2.11 Φ is automatically bounded on any finite interval; and second, our only lower candidate, $b(x) = a_f(\sqrt{x_1 x_2})$, is continuous on Ω_Q .

We now turn to finding candidates B and b satisfying the hypotheses of Lemma 4.2. Let us focus on the upper candidate B ; for the lower candidate, the steps below are exactly the same, except all inequality signs are reversed.

If B is sufficiently differentiable, condition (4.6) yields the following differential inequality:

$$(4.10) \quad \frac{1}{8} [dx_1 \ dx_2] d^2 B(x) [dx_1 \ dx_2]^T + \Phi(x_1 x_2) \left[\left(\frac{dx_1}{x_1} \right)^2 + \left(\frac{dx_2}{x_2} \right)^2 \right] \leq 0,$$

where $d^2 B$ is the Hessian of B .

In matrix form, we get

$$(4.11) \quad \begin{bmatrix} B_{x_1 x_1} + 8 \frac{\Phi(x_1 x_2)}{x_1^2} & B_{x_1 x_2} \\ B_{x_1 x_2} & B_{x_2 x_2} + 8 \frac{\Phi(x_1 x_2)}{x_2^2} \end{bmatrix} \leq 0.$$

The best (true) candidate B must accommodate the existence of an optimizing weight w (or a sequence thereof), which would produce an equality (or approximate equality) on every step of the inductive process of Lemma 4.2. In problems that admit infinitesimal forms such as (4.11) this typically means that the kernel of the corresponding differential matrix is non-trivial. Imposing this condition on B , we obtain the differential equation

$$(4.12) \quad \left(B_{x_1 x_1} + 8 \frac{\Phi(x_1 x_2)}{x_1^2} \right) \left(B_{x_2 x_2} + 8 \frac{\Phi(x_1 x_2)}{x_2^2} \right) = B_{x_1 x_2}^2, \quad x \in \Omega_Q.$$

We couple this equation with the boundary condition

$$(4.13) \quad B(x_1, 1/x_1) = 0$$

and, to ensure that $d^2 B \leq 0$, with the inequalities

$$(4.14) \quad B_{x_1 x_1} + 8 \frac{\Phi(x_1 x_2)}{x_1^2} \leq 0; \quad B_{x_2 x_2} + 8 \frac{\Phi(x_1 x_2)}{x_2^2} \leq 0.$$

We now solve the system (4.12), (4.13) using the homogeneity statement (4.4) from Lemma 4.1. Recall our notation:

$$s = \sqrt{x_1 x_2}, \quad f(s) = \Phi(s^2), \quad L = \sqrt{Q}.$$

Let $\mathcal{A}(s)$ stand for either $B(s, s)$ or $b(s, s)$, as needed. With this notation, and upon differentiation, (4.12) and (4.13) become

$$(4.15) \quad \left(s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32 f(s) \right)^2 = \left(s^2 \mathcal{A}''(s) + s \mathcal{A}'(s) \right)^2, \quad 1 \leq s \leq L, \quad \mathcal{A}(1) = 0.$$

The conditions (4.14) and their equivalent for b become

$$(4.16) \quad s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32 f(s) \leq 0 \quad \text{and} \quad s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32 f(s) \geq 0,$$

respectively. Taking square roots in (4.15), we obtain two elementary linear equations:

$$\mathcal{A}'(s) = 16 \frac{f(s)}{s}; \quad \mathcal{A}''(s) = -16 \frac{f(s)}{s^2}.$$

Accordingly, we have two solutions, one of which is completely determined by the initial condition $\mathcal{A}(1) = 0$ and the other contains a parameter:

$$(4.17) \quad a_f(s) = 16 \int_1^s \frac{f(r)}{r} dr$$

and

$$(4.18) \quad A_f(s) = 16 \left[\int_1^s \frac{f(r)}{r} dr - s \int_1^s \frac{f(r)}{r^2} dr \right] + C(s-1).$$

How do we determine the constant C ? In general, this would depend on whether we intend for A_f to be an upper or lower Bellman candidate, and on what assumptions are made on f . In this paper, we only use A_f as an upper candidate and so we want C to be the smallest number such that the first inequality in (4.16) is satisfied on $[1, L]$. For $\mathcal{A} = A_f$ that inequality is equivalent to

$$C \geq 16 \left[\frac{f(s)}{s} + \int_1^s \frac{f(z)}{z^2} dz \right], \quad 1 \leq s \leq L.$$

If f is increasing, as it is assumed to be in Part (1) of Theorem 2.10, the maximum of the right-hand side is attained when $s = L$, thus we set

$$C = 16 \left[\frac{f(L)}{L} + \int_1^L \frac{f(z)}{z^2} dz \right],$$

which is equivalent to setting $L^2 A_f''(L) - L A_f'(L) + 32 f(L) = 0$. This yields a complete Bellman candidate

$$(4.19) \quad A_{L,f}(s) = 16 \left[\frac{f(L)}{L} + \int_1^L \frac{f(z)}{z^2} dz \right] (s-1) - 16 \int_1^s \frac{f(z)}{z^2} (s-z) dz.$$

In the next section, we verify that the functions $a_f(\sqrt{x_1 x_2})$ and $A_{L,f}(\sqrt{x_1 x_2})$ satisfy the appropriate “main inequalities” – either (4.6) or (4.7), depending on the assumptions on f .

5. VERIFICATION OF THE MAIN INEQUALITY

Lemma 4.2 says that if a Bellman candidate satisfies the main inequality, (4.6) or (4.7), and the boundary condition (4.8), then it automatically gives an upper or lower estimate on the Bellman function itself. This section is devoted to verifying the main inequality(ies) for the candidates a_f and $A_{L,f}$ (which satisfy the boundary condition by construction) and thus proving the following lemma, which establishes the upper estimates in Parts (1) and (3) of Theorem 2.10 and the lower estimate in Part (1) of Theorem 2.11.

Lemma 5.1. *Let $L = \sqrt{Q}$, $f(z) = \Phi(z^2)$, and $h(z) = f(z)/z^2$.*

(1) *If Φ is increasing and h is convex, then*

$$B_{Q,\Phi}(x_1, x_2) \leq A_{L,f}(\sqrt{x_1 x_2}).$$

(2) *If Φ is decreasing, then*

$$B_{Q,\Phi}(x_1, x_2) \leq a_f(\sqrt{x_1 x_2}).$$

(3) *If h is increasing, then*

$$b_{Q,\Phi}(x_1, x_2) \geq a_f(\sqrt{x_1 x_2}).$$

The proofs of the three separate parts of this lemma are contained in Lemmas 5.2, 5.5, and 5.6. First, let us rephrase (4.6) and (4.7) using the homogeneity of the problem. Suppose B is an upper candidate. Let $\mathcal{A}(s) = B(s, s)$. Alongside $s = \sqrt{x_1 x_2}$, we will use $s^\pm = \sqrt{x_1^\pm x_2^\pm}$. A simple calculation shows that

$$\frac{(x_1^- - x_1^+)^2}{x_1^2} + \frac{(x_2^- - x_2^+)^2}{x_2^2} = 8 - 4 \frac{(s^-)^2 + (s^+)^2}{s^2} + \frac{((s^-)^2 - (s^+)^2)^2}{s^4},$$

and so (4.6) is equivalent to

$$(5.1) \quad \mathcal{A}(s) - \frac{1}{2}\mathcal{A}(s^-) - \frac{1}{2}\mathcal{A}(s^+) - f(s) \left[8 - 4 \frac{(s^-)^2 + (s^+)^2}{s^2} + \frac{((s^-)^2 - (s^+)^2)^2}{s^4} \right] \geq 0.$$

If b is a lower candidate, and $\mathcal{A}(s) = b(s, s)$, then the inequality to be verified, (4.7), becomes

$$(5.2) \quad \mathcal{A}(s) - \frac{1}{2}\mathcal{A}(s^-) - \frac{1}{2}\mathcal{A}(s^+) - f(s) \left[8 - 4 \frac{(s^-)^2 + (s^+)^2}{s^2} + \frac{((s^-)^2 - (s^+)^2)^2}{s^4} \right] \leq 0.$$

It is important to determine the domain of these variables. Clearly $(s^-, s^+, s) \in [1, L]^3$, but this is too crude. If s^- and s^+ are fixed, s cannot be too small; this is reasonably clear from the geometry of the problem. Formally, to find the smallest s we solve the following problem

$$\min \left\{ \left(\frac{x_1^- + x_1^+}{2} \right) \left(\frac{x_2^- + x_2^+}{2} \right) : x_1^- x_2^- = (s^-)^2, x_1^+ x_2^+ = (s^+)^2 \right\}.$$

It is easy to check that this minimum is attained when all three points x^-, x^+ , and $(x^- + x^+)/2$ lie on the same line through the origin. In this case, $s = \sqrt{x_1 x_2} = (s^- + s^+)/2$. Thus the domain over which (5.1) or (5.2) needs to be verified for each particular choice of \mathcal{A} is

$$(5.3) \quad \omega_L \stackrel{\text{def}}{=} \left\{ (s^-, s^+, s) : 1 \leq s^- \leq L; \quad 1 \leq s^+ \leq L; \quad \frac{s^- + s^+}{2} \leq s \leq L \right\}.$$

Consider a new function on ω_L :

$$(5.4) \quad P(s^-, s^+, s) = \mathcal{A}(s) - \frac{1}{2}\mathcal{A}(s^-) - \frac{1}{2}\mathcal{A}(s^+) - f(s) \left[8 - 4 \frac{(s^-)^2 + (s^+)^2}{s^2} + \frac{((s^-)^2 - (s^+)^2)^2}{s^4} \right].$$

In the lemmas below, we will set either $\mathcal{A} = A_{L,f}$ or a_f , depending on the conditions assumed on f and on whether we are proving \mathcal{A} to be an upper Bellman candidate, in which case we need to show that $P \geq 0$ on ω_L , or a lower one, for which we need to show $P \leq 0$. We split further presentation in this section in three parts.

5.1. $A_{L,f}$ as an upper candidate. According to the discussion above, we have to show that if f is increasing and h is convex, then P given by (5.4) with $\mathcal{A} = A_{L,f}$ is non-negative on the domain ω_L . We will prove a somewhat stronger statement, and one that is much easier to handle computationally. Let

$$(5.5) \quad U(s^-, s^+, s) = A_{L,f}(s) - \frac{1}{2}A_{L,f}(s^-) - \frac{1}{2}A_{L,f}(s^+) - 8f(s) \left[1 - \frac{s^- s^+}{s^2} \right].$$

Observe that $P \geq U$ on ω_L and $P(s^-, s^+, \frac{s^- + s^+}{2}) = U(s^-, s^+, \frac{s^- + s^+}{2})$.

Lemma 5.2. *If f is a non-negative, increasing function on $[1, \infty)$ and $h(z) = f(z^2)/z^2$ is convex, then $U \geq 0$ on ω_L .*

Proof. Throughout the proof, we will write A for $A_{L,f}$. Let us collect a couple of useful facts about A . A direct calculation gives

$$A'(s) = 16 \left[\frac{f(L)}{L} + \int_s^L h(z) dz \right].$$

Since f is increasing, we have

$$(5.6) \quad A'(s) \geq 16 \left[\frac{f(L)}{L} + f(s) \int_s^L \frac{1}{z^2} dz \right] = 16 \frac{f(s)}{s}.$$

In addition, since h is convex, we have, for any $1 \leq s_1 \leq s_2 \leq L$,

$$(5.7) \quad A'(s_1) - A'(s_2) = 16 \int_{s_1}^{s_2} h(z) dz \geq 16(s_2 - s_1) h\left(\frac{s_1 + s_2}{2}\right).$$

In the next two lemmas we first reduce the inequality $U \geq 0$ on ω_L to two of its special cases and then verify them.

Lemma 5.3. $U \geq 0$ on ω_L if and only if

$$(5.8) \quad U\left(s_1, s_2, \frac{s_1 + s_2}{2}\right) \geq 0 \quad \text{for all } s_1, s_2 \in [1, L]$$

and

$$(5.9) \quad U(s_1, s_1, s_2) \geq 0 \quad \text{for all } 1 \leq s_1 \leq s_2 \leq L.$$

Proof. Both (5.8) and (5.9) are clearly necessary. To show the sufficiency, take any point $(s^-, s^+, s) \in \omega_L$. Assume, without loss of generality, that $s^- \leq s^+$; then either $s^- \leq s \leq s^+$ or $s^- \leq s^+ < s$. Let us consider these cases separately.

The case $s^- \leq s \leq s^+$. Since $s \geq (s^- + s^+)/2$, we have $s^- \leq 2s - s^+ \leq s \leq s^+$. Using (5.6) and (5.7), we have

$$A'(s^-) \geq A'(2s - s^+) \geq A'(s^+) + 32(s^+ - s)h(s) \geq 16 \frac{f(s^+)}{s^+} + 32(s^+ - s)h(s).$$

Differentiating U with respect to s^- , and noting that $f(s^+) \geq f(s)$, we get

$$\begin{aligned} \frac{\partial U}{\partial s^-} &= -\frac{1}{2}A'(s^-) + 8 \frac{f(s)}{s^2} s^+ \\ &\leq -8 \frac{f(s^+)}{s^+} - 16(s^+ - s) \frac{f(s)}{s^2} + 8 \frac{f(s)}{s^2} s^+ \\ &\leq -8 \frac{f(s)}{s^+} - 16(s^+ - s) \frac{f(s)}{s^2} + 8 \frac{f(s)}{s^2} s^+ \\ &= -8 \frac{f(s)}{s^+ s^2} (s^+ - s)^2 \leq 0. \end{aligned}$$

Therefore,

$$U(s^-, s^+, s) \geq U(2s - s^+, s^+, s),$$

which is positive by (5.8).

The case $s^- \leq s^+ < s$. We have

$$\begin{aligned} \frac{\partial U}{\partial s^-} &= -\frac{1}{2}A'(s^-) + 8\frac{f(s)}{s^2}s^+ \\ &\leq -\frac{1}{2}A'(s) + 8\frac{f(s)}{s^2}s^+ \\ &\leq -8\frac{f(s)}{s} + 8\frac{f(s)}{s^2}s^+ \\ &= -8\frac{f(s)}{s^2}(s - s^+) \leq 0. \end{aligned}$$

Therefore,

$$U(s^-, s^+, s) \geq U(s^+, s^+, s),$$

which is positive by (5.9). □

It remains to verify (5.8) and (5.9).

Lemma 5.4. *Under the assumptions of Lemma 5.2, inequalities (5.8) and (5.9) hold.*

Proof. To prove (5.8), we need to show that for all $s_1, s_2 \in [1, L]$,

$$A\left(\frac{s_1 + s_2}{2}\right) - \frac{1}{2}(A(s_1) + A(s_2)) - 2h\left(\frac{s_1 + s_2}{2}\right)(s_2 - s_1)^2 \geq 0.$$

It is a simple exercise to verify that if $s \in \mathbb{R}$, $\Delta \geq 0$, and u is a twice-differentiable function on the interval $[s - \Delta, s + \Delta]$, then

$$\begin{aligned} u(s) - \frac{1}{2}(u(s - \Delta) + u(s + \Delta)) &= -\frac{1}{2} \int_{-\Delta}^{\Delta} (\Delta - |t|) u''(s + t) dt \\ &= -\frac{1}{2} \int_{-\Delta}^{\Delta} (\Delta - |t|) \left(\frac{1}{2} u''(s + t) + \frac{1}{2} u''(s - t) \right) dt. \end{aligned}$$

Using this formula with $u = A$, $u'' = -16h$, $s = \frac{s_1 + s_2}{2}$, and $\Delta = |s_2 - s_1|/2$, we have

$$\begin{aligned} (5.10) \quad A\left(\frac{s_1 + s_2}{2}\right) - \frac{1}{2}(A(s_1) + A(s_2)) - 2h\left(\frac{s_1 + s_2}{2}\right)(s_2 - s_1)^2 \\ = 8 \int_{-\Delta}^{\Delta} (\Delta - |t|) \left(\frac{1}{2} h(s + t) + \frac{1}{2} h(s - t) - h(s) \right) dt \geq 0, \end{aligned}$$

where the last inequality follows because h is convex.

To prove (5.9), we have to show that

$$A(s_2) - A(s_1) \geq 8f(s_2) \left[1 - \left(\frac{s_1}{s_2} \right)^2 \right].$$

Since A' is decreasing and $A'(s) \geq 16f(s)/s$, we have

$$A(s_2) - A(s_1) = \int_{s_1}^{s_2} A'(z) dz \geq A'(s_2)(s_2 - s_1) \geq 16 \frac{f(s_2)}{s_2} (s_2 - s_1) \geq 8f(s_2) \left[1 - \left(\frac{s_1}{s_2} \right)^2 \right]$$

□

The proof of Lemma 5.2 is now complete. □

5.2. a_f as an upper candidate. Here, we have to show that if f is a decreasing function, then P given by (5.4) with $\mathcal{A} = a_f$ satisfies $P \geq 0$ on ω_L . As in the previous case, we instead consider the function

$$V(s^-, s^+, s) = a_f(s) - \frac{1}{2}a_f(s^-) - \frac{1}{2}a_f(s^+) - 8f(s) \left[1 - \frac{s^- s^+}{s^2} \right].$$

In addition, we will slightly expand the domain. Let

$$(5.11) \quad \omega_\infty = \left\{ (s^-, s^+, s) : s^- \geq 1, \quad s^+ \geq 1, \quad s \geq \frac{s^- + s^+}{2} \right\}.$$

Observe that $\omega_L \subset \omega_\infty$ and $P \geq V$ on ω_∞ .

Lemma 5.5. *If f is a non-negative, decreasing function on $[1, \infty)$, then $V \geq 0$ on ω_∞ .*

Proof. The definition of a_f gives

$$V(s^-, s^+, s) = 8 \int_{s^-}^s \frac{f(r)}{r} dr + 8 \int_{s^+}^s \frac{f(r)}{r} dr - 8f(s) \left[1 - \frac{s^- s^+}{s^2} \right].$$

Since f is decreasing, we have

$$\int_{s^-}^s \frac{f(r)}{r} dr \geq f(s) \log \left(\frac{s}{s^-} \right), \quad \int_{s^+}^s \frac{f(r)}{r} dr \geq f(s) \log \left(\frac{s}{s^+} \right)$$

(note that one of these integrals may be negative, but the inequality still holds), and so

$$V(s^-, s^+, s) \geq 8f(s) \left[\log \left(\frac{s^2}{s^- s^+} \right) - 1 + \frac{s^- s^+}{s^2} \right].$$

Since $\frac{s^- s^+}{s^2} \leq 1$, we conclude that $V(s^-, s^+, s) \geq 0$. \square

5.3. a_f as a lower candidate. Here, we show that if h is an increasing function, then the function P given by (5.4) with $\mathcal{A} = a_f$ satisfies $P \leq 0$ on ω_L . We will again prove a slightly stronger result. Namely, let

$$W(s^-, s^+, s) = a_f(s) - \frac{1}{2}a_f(s^-) - \frac{1}{2}a_f(s^+) - 4f(s) \left[2 - \frac{(s^-)^2 + (s^+)^2}{s^2} \right].$$

Observe that $P \leq W$ on the domain ω_∞ given by (5.11) and that $P(s^+, s^+, s) = W(s^+, s^+, s)$.

Lemma 5.6. *If f is a non-negative function on $[1, \infty)$ such that $h(s) = f(s)/s^2$ is increasing, then $W \leq 0$ on ω_∞ .*

Proof. The proof is similar to that of Lemma 5.5. We have

$$W(s^-, s^+, s) = 8 \int_{s^-}^s \frac{f(r)}{r} dr + 8 \int_{s^+}^s \frac{f(r)}{r} dr - 4f(s) \left[2 - \frac{(s^-)^2 + (s^+)^2}{s^2} \right].$$

Since h is increasing, we can write

$$\int_{s^-}^s \frac{f(r)}{r} dr = \int_{s^-}^s \frac{f(r)}{r^2} r dr \leq \frac{f(s)}{s^2} \int_{s^-}^s r dr = \frac{1}{2} f(s) \left(1 - \frac{(s^-)^2}{s^2} \right),$$

and similarly for the second integral, which immediately gives $W(s^-, s^+, s) \leq 0$. \square

6. THE CASE OF CONCAVE h : PROOF OF THE MAIN ESTIMATE AND DISCUSSION

In the case when the function $h(s) = \Phi(s^2)/s^2$ is concave, we do not obtain the exact Bellman functions $B_{Q,\Phi}$. Instead, we derive good pointwise estimates for $B_{Q,\Phi}$ using the simple observation that the graph of h lies below all of its tangents (since h is concave, it does have one-sided tangents at every point). Thus, we majorate h by its tangent at a point $s = s_0$, use Part (1) of Lemma 5.1 in conjunction with this convex majorant, and then optimize over all s_0 .

To make presentation smoother, and to illustrate in what sense the resulting estimate can be construed as optimal, the derivation below assumes that h is differentiable. However, the final formula does not use derivatives, but only the fact that for every $s_0 \in (1, L]$, there exists a number $m(s_0)$ such that $h(s) \leq m(s_0)(s - s_0) + h(s_0)$. Accordingly, we could obtain the same result by replacing all instances of $h'(s_0)$ with $m(s_0)$ and then simply taking the value of s_0 given by (6.1) below.

Proof of Theorem 2.10, Part (2). Without loss of generality, assume $L > 1$. Fix $s_0 \in (1, L]$ and let $h_{s_0}(s) = h'(s_0)(s - s_0) + h(s_0)$ be the tangent to the graph of h at s_0 . Let $f_{s_0}(s) = s^2 h_{s_0}(s)$; then $f(s) = s^2 h(s) \leq f_{s_0}(s)$ or, equivalently, $\Phi(s^2) \leq \Phi_{s_0}(s^2) \stackrel{\text{def}}{=} f_{s_0}(s)$. Therefore, for any $w \in A_2(Q)$,

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) \leq \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{\Phi_{s_0}}(w),$$

which means that

$$B_{Q,\Phi} \leq B_{Q,\Phi_{s_0}}.$$

Since h_{s_0} is convex, we can estimate the Bellman function on the right using Part (1) of Lemma 5.1:

$$B_{Q,\Phi_{s_0}}(x_1, x_2) \leq A_{L,f_{s_0}}(\sqrt{x_1 x_2}),$$

where $A_{L,f}$ is given by (2.5). An easy calculation yields

$$A_{L,f_{s_0}}(s) = \frac{8}{3} (s - 1) [3(h(s_0) - s_0 h'(s_0))(4L - s - 1) + h'(s_0)(9L^2 - s^2 - s - 1)].$$

To minimize this expression with respect to s_0 , we compute

$$\frac{\partial A_{L,f_{s_0}}(s)}{\partial s_0} = \frac{8}{3} (s - 1) h''(s_0) [9L^2 - s^2 - s - 1 - s_0(4L - s - 1)].$$

Since h is concave (note that we do not need strict concavity here) and $4L - s - 1 \geq 0$, the minimum is attained at

$$(6.1) \quad s_0(s) = \frac{9L^2 - s^2 - s - 1}{3(4L - s - 1)}.$$

It is easy to verify that $s_0 \in (1, L]$. Plugging this back into $A_{L,f_{s_0}}$, we get

$$(6.2) \quad A_{L,f_{s_0(s)}}(s) = 8(s - 1)(4L - s - 1) h(s_0(s)),$$

which completes the proof. \square

Let us briefly discuss this result. First, we note that the optimal tangent does not depend on the choice of h , making formula (6.2) particularly easy to use. This estimate also demonstrates the utility of our sharp estimates for convex h , proved in Lemma 5.1 of the previous section. As noted earlier, and as is clear from the statement of Corollary 2.13, for power functions $\Phi(t) = t^\alpha$, $1 \leq \alpha \leq 3/2$, the estimate $B_{Q,f}(x_1, x_2) \leq A_{L,f_{s_0}}(\sqrt{x_1 x_2})$ amounts to writing

$$f(s) = s^{2\alpha} = (s^2)^{3-2\alpha} (s^3)^{2\alpha-2}$$

and then using Hölder's inequality to interpolate between the sharp results for $\alpha = 1$ and $\alpha = 3/2$. This connection is not apparent from the general formula (6.2).

However, the estimate just proved is not sharp; here is one way to improve it. We have

$$\frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^\Phi(w) = \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{\Phi_{s_0}}(w) - \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} c_J^{\Phi_{s_0} - \Phi}(w).$$

The function $\tilde{\Phi} \stackrel{\text{def}}{=} \Phi_{s_0} - \Phi$ is non-negative, so we can formally write

$$(6.3) \quad \mathbf{B}_{Q,\Phi} \leq \mathbf{B}_{Q,\Phi_{s_0}} - \mathbf{b}_{Q,\tilde{\Phi}}.$$

Theorem 2.11 gives the formula for lower Bellman functions $\mathbf{b}_{Q,\Phi}$ when $h(s) = \Phi(s^2)/s^2$ is increasing on $[1, \infty)$. However, $\tilde{\Phi}(s^2)/s^2$ does not have this property – in fact, it is decreasing for $s \leq s_0$ and increasing for $s \geq s_0$. We could find the Bellman function for $(\Phi - \Phi_{s_0})\chi_{\{s \geq s_0\}}$ as a substitute, but, as a willing reader can verify, the resulting formula would be much more cumbersome than (6.2) and the improvement in the estimate, numerically quite small. In any case, such manipulations cannot produce a sharp estimate, because, as we demonstrate in Section 7 below, the optimizing weights for the Bellman candidates $A_{L,f}$ and a_f are essentially unique – and different. This implies that an estimate of the form (6.3), combining sharp estimates that involve both $A_{L,f}$ and a_f , cannot itself be sharp.

If one desires, as we do, to find the actual function $\mathbf{B}_{Q,\Phi}$ in the case when h is concave, one has to understand the nature of the extremizing split $x = (x^- + x^+)/2$ in the main inequality (4.1), i.e., the choice of x^\pm that turns the inequality into an equality or approximate equality. The candidates $A_{L,f}$ and a_f were derived under the assumption that such x^\pm are infinitesimally close to x , meaning (4.6) is equivalent to its infinitesimal version, (4.10). However, for concave h , $A_{L,f}$ fails the main inequality at every point, as is clear from formula (5.10) in the proof of Lemma 5.1. For similar reasons, a_f does not work either. Therefore, the optimizing split is not infinitesimal in this case.

An approach that seems promising is to consider piecewise-linear concave h , and then approximate an arbitrary concave h by such functions. One then would look for the Bellman candidate \mathcal{A} such that

$$\mathcal{A}''(s) = -16h(s) + \sum_k c_k \delta_{s_k}(s),$$

where s_k are the points of discontinuity of h' , δ_{s_k} are Dirac masses at s_k , and c_k are the negative coefficients chosen so that the main inequality for the resulting candidate \mathcal{A} is satisfied on the whole domain. The presence of c_k would entail a non-infinitesimal optimal split at the point $x = (s_k, s_k)$. The nature of the dependence of the values of c_k on the local behavior of h is the subject of further study.

7. OPTIMIZERS

In this section we prove the converse inequalities to those established in Lemma 5.1, thus completing the proof of Theorems 2.10 and 2.11. This is done through the construction of two optimizing sequences – one for a_f and one for $A_{L,f}$. Importantly, these sequences do not depend on f , meaning that the upper Bellman function $\mathbf{B}_{Q,\Phi}$ is linear with respect to Φ for all increasing Φ such that $h(s) = \Phi(s^2)/s^2$ is convex and, separately, for all decreasing Φ . Likewise, the lower Bellman function $\mathbf{b}_{Q,\Phi}$ is linear in Φ on the class of all Φ such that h is increasing.

Without loss of generality, we will define our optimizers almost everywhere on $I = (0, 1)$. Fix a point $x \in \Omega_Q$. We say that a sequence of functions $\{w_n^x\}$ on $(0, 1)$ is an *optimizing*

sequence for a Bellman candidate \mathcal{A} at x , if $\{w_n^x\}$ satisfies the following three conditions:

$$(7.1) \quad \forall n, \quad w_n^x \in A_2^{d,Q}((0,1));$$

$$(7.2) \quad \forall n, \quad \langle w_n^x \rangle_{(0,1)} = x_1, \langle (w_n^x)^{-1} \rangle_{(0,1)} = x_2;$$

$$(7.3) \quad \sum_{J \in \mathcal{D}(0,1)} |J| \Phi \left(\langle w_n^x \rangle_J \langle (w_n^x)^{-1} \rangle_J \right) R_J(w_n^x) \longrightarrow \mathcal{A}(\sqrt{x_1 x_2}), \text{ as } n \rightarrow \infty.$$

In some settings, we are lucky to have an actual optimizing function, meaning that w_n^x are the same for all n , but it is rare in dyadic problems. The recursive construction of optimizers in this section is similar to the one used in [19] and [16].

Observe that it follows directly from the definitions (2.3) and (2.4) that if conditions (7.1)-(7.3) are satisfied, then

$$\begin{aligned} B_{Q,\Phi}(x) &\geq \mathcal{A}(\sqrt{x_1 x_2}), \\ b_{Q,\Phi}(x) &\leq \mathcal{A}(\sqrt{x_1 x_2}). \end{aligned}$$

Note that due to the homogeneity of the problem it suffices to find $\{w_n^x\}$ only on the line $x_1 = x_2$. Indeed, if $\{w_n^{(s,s)}\}$ is an optimizer for the point (s, s) , then $\{\tau w_n^{(s,s)}\}$ is an optimizer for the point $(\tau s, \tau^{-1} s)$, and we can simply set $s = \sqrt{x_1 x_2}$ and $\tau = \sqrt{x_1/x_2}$.

The optimizers for a_f and $A_{L,f}$ are different, but each construction starts with the vector field generated by the kernel of the matrix in (4.11). Specifically, replace $B(x_1, x_2)$ in (4.11) by $\mathcal{A}(\sqrt{x_1 x_2})$. This gives the equivalent matrix

$$\begin{bmatrix} \frac{1}{x_1^2} (s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32f(s)) & \mathcal{A}''(s) + \mathcal{A}'(s) \frac{1}{s} \\ \mathcal{A}''(s) + \mathcal{A}'(s) \frac{1}{s} & \frac{1}{x_2^2} (s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32f(s)) \end{bmatrix},$$

whose kernel is given by

$$(7.4) \quad (s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32f(s)) x_2 dx_1 + (s^2 \mathcal{A}''(s) - s \mathcal{A}'(s)) x_1 dx_2 = 0.$$

We now split the presentation in two parts.

7.1. The optimizer for a_f . When $\mathcal{A} = a_f$, we have $\mathcal{A}'(s) = 16f(s)/s$ and so (7.4) becomes

$$f'(s)(x_2 dx_1 + x_1 dx_2) = 0,$$

provided f' is defined at s . At points x with $f'(\sqrt{x_1 x_2}) = 0$ or $f'(\sqrt{x_1 x_2})$ is undefined, (7.4) does not give any information. However, when $f' \neq 0$, we get

$$(7.5) \quad \frac{dx_2}{dx_1} = -\frac{x_2}{x_1},$$

meaning that the vector field consists, locally, of tangents to hyperbolas of the form $x_1 x_2 = C$.

Now, fix $s \in [1, L]$ and $n \in \mathbb{N}$. For $k = 0, \dots, n$, let $s_k = \sqrt{s^2(1 - k/n) + k/n}$; also let $\Delta_n = \sqrt{(s^2 - 1)/n}$ so that $s_{k+1}^2 = s_k^2 - \Delta_n^2$. We will define $w_n^{(s,s)}$ as a function that is constant on each of the 2^{-n} intervals in $\mathcal{D}_n((0,1))$. In what follows, let us write $w_n^{(k)}$ for $w_n^{(s_k, s_k)}$; in this notation $w_n^x = w_n^{(s,s)} = w_n^{(0)}$.

We start by splitting the point $x = (s, s)$ into x^- and x^+ along the tangent vector field (7.5): let $x^\pm = (s \pm \Delta_n, s \mp \Delta_n)$ (note: the larger the n , the more infinitesimal the

split). Define w_n^x to be the following concatenation of $w_n^{x^-}$ and $w_n^{x^+}$:

$$w_n^x(t) = \begin{cases} w_n^{x^-}(2t), & t \in (0, 1/2), \\ w_n^{x^+}(2t-1), & t \in (1/2, 1). \end{cases}$$

Due to homogeneity, set $w_n^{x^\pm} = \sqrt{\frac{s \pm \Delta_n}{s \mp \Delta_n}} w_n^{(1)}$; equivalently, $w_n^{x^\pm} = \frac{s_1}{s \mp \Delta_n} w_n^{(1)}$. Repeat this process $n-1$ more times, on each step setting

$$(7.6) \quad w_n^{(k)}(t) = \begin{cases} \frac{s_{k+1}}{s_k + \Delta_n} w_n^{(k+1)}(2t), & t \in (0, 1/2), \\ \frac{s_{k+1}}{s_k + \Delta_n} w_n^{(k+1)}(2t-1), & t \in (1/2, 1). \end{cases}$$

After n steps, we will have $s_n = 1$ and, since the only test functions on the boundary are constants, we have to set

$$(7.7) \quad w_n^{(n)}(t) = 1, \quad t \in (0, 1).$$

This completely defines w_n^x . The construction is pictured in Figure 1.

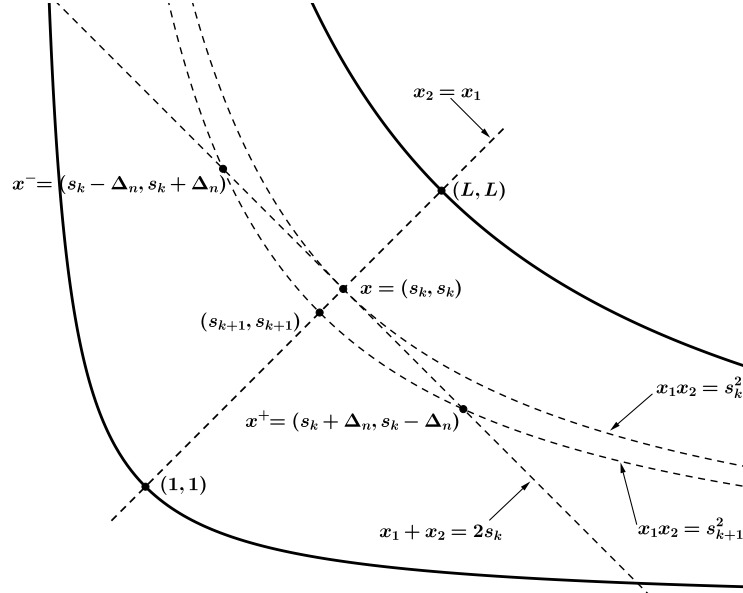


FIGURE 1. The optimizer for a_f : tangential splits $x = \frac{1}{2}(x^- + x^+)$.

Our main result for this section now follows.

Lemma 7.1. *The sequence $\{w_n^{(s,s)}\}$ defined by (7.6) and (7.7) is an optimizing sequence for a_f at (s, s) for all $s \in [1, L]$. Consequently,*

(i) *If Φ is decreasing, then*

$$\mathbf{B}_{Q,\Phi}(x) \geq a_f(\sqrt{x_1 x_2}), \quad \forall x \in \Omega_Q.$$

(ii) *If h is increasing, then*

$$\mathbf{b}_{Q,\Phi}(x) \leq a_f(\sqrt{x_1 x_2}), \quad \forall x \in \Omega_Q.$$

Proof. The verification of (7.1) and (7.2) consists of a (backward) finite induction on k : first consider $w_n^{(n-1)}$ and check that $\langle w_n^{(n-1)} \rangle_I = s_{n-1}$, $\langle (w_n^{(n-1)})^{-1} \rangle_I = s_{n-1}$, and $w_n^{(n-1)} \in A_2^{d,Q}(I)$; then do the same for $w_n^{(n-2)}$, and so on. We leave the details to the reader.

It remains to verify (7.3). We have

$$\sum_{J \in \mathcal{D}(0,1)} |J| \Phi \left(\langle w_n^x \rangle_J \langle (w_n^x)^{-1} \rangle_J \right) R_J(w_n^x) = \sum_{k=0}^{n-1} \sum_{J \in \mathcal{D}_k(0,1)} 2^{-k} \Phi \left(\langle w_n^{x_J} \rangle_J \langle (w_n^{x_J})^{-1} \rangle_J \right) R_J(w_n^{x_J}),$$

where $w_n^{x_J}$ is completely determined by the generation of J and its position among fellow intervals of the same generation. In fact, for any two intervals I and J of the k -th generation, $w_n^{x_I}$, $w_n^{x_J}$, and $w_n^{(k)}$ differ only by a multiplicative factor, which does not affect their contributions to the overall sum. Thus,

$$\begin{aligned} & \sum_{k=0}^{n-1} \sum_{J \in \mathcal{D}_k(0,1)} 2^{-k} \Phi \left(\langle w_n^{x_J} \rangle_J \langle (w_n^{x_J})^{-1} \rangle_J \right) R_J(w_n^{x_J}) \\ &= \sum_{k=0}^{n-1} \sum_{J \in \mathcal{D}_k(0,1)} 2^{-k} \Phi \left(\langle w_n^{(k)} \rangle_J \langle (w_n^{(k)})^{-1} \rangle_J \right) R_J(w_n^{(k)}) \\ &= \sum_{k=0}^{n-1} \Phi(s_k \cdot s_k) \left[\left(\frac{(s_k - \Delta_n) - (s_k + \Delta_n)}{s_k} \right)^2 + \left(\frac{(s_k + \Delta_n) - (s_k - \Delta_n)}{s_k} \right)^2 \right] \\ &= 8 \sum_{k=0}^{n-1} \Phi(s_k^2) \frac{\Delta_n^2}{s_k^2} \xrightarrow{n \rightarrow \infty} 8 \int_1^{s^2} \frac{\Phi(t)}{t} dt = 16 \int_1^s \frac{f(t)}{t} dt = a_f(s). \end{aligned}$$

□

Remark 7.2. The optimizer $\{w_n\}$ from (7.6) and (7.7) has one more property, which is used in the proof of Theorem 2.3 in Section 3. Namely, for any $R \in \mathcal{D}_k(0,1)$,

$$\frac{1}{|R|} \sum_{J \in \mathcal{D}(R)} c_J^\Phi(w_n) = \sum_{j=0}^{k-1} \Phi(s_j^2) \frac{\Delta_n^2}{s_j^2} \leq \sum_{J \in \mathcal{D}(0,1)} c_J^\Phi(w_n).$$

Therefore, if one considers the “local” Carleson norm of $\{c_J^\Phi(w)\}$, by taking the supremum in (1.1) over all dyadic subintervals of $(0,1)$, as opposed to all dyadic intervals in \mathcal{D} , that norm is realized on the the interval $(0,1)$ itself.

7.2. The optimizer for $A_{L,f}$. When $\mathcal{A} = A_{L,f}$, given by (4.19), we have $s^2 \mathcal{A}''(s) - s \mathcal{A}'(s) + 32f(s) = -(s^2 \mathcal{A}''(s) + s \mathcal{A}'(s))$ and (7.4) becomes

$$(s^2 \mathcal{A}''(s) + s \mathcal{A}'(s))(x_2 dx_1 - x_1 dx_2) = 0.$$

A simple calculation shows that $s^2 \mathcal{A}''(s) + s \mathcal{A}'(s) = 16 \left(f(s) + f(L) \frac{s}{L} + \int_s^L h(z) dz \right)$, meaning that if f is monotone and non-constant, then the only point where $s^2 \mathcal{A}''(s) + s \mathcal{A}'(s) = 0$ is $s = L$. This gives no information as to the direction in which we should split a point on the boundary. However, infinitesimally, only tangential splitting will keep the new points in the domain, so that will be our choice. At interior points of Ω_Q we have

$$(7.8) \quad \frac{dx_2}{dx_1} = \frac{x_2}{x_1},$$

meaning that the vector field is locally aligned with lines through the origin, $x_2 = Cx_1$.

As before, homogeneity allows us to construct an optimizing sequence only along the segment $x_1 = x_2 = s, 1 \leq s \leq L$. Furthermore, to save effort we will construct an optimizing sequence for the point $x = (s, s)$ only when s is a dyadically-rational point in the interval $[1, L]$ and then use an approximation argument to establish our key result. Thus, let $s = L - m2^{-N}(L - 1)$ for some integer $N \geq 0$ and an odd $m \in \{0, 1, \dots, 2^N\}$.

Take $n \geq 1$ and let $\Delta_n = 2^{-N-n}(L - 1)$, $s_k = L - k\Delta_n$, $k = 0, 1, \dots, 2^{N+n}$. In this notation, $s = s_{m2^n}$. Similarly to the previous case, defining an optimizing sequence for the point (s_{m2^n}, s_{m2^n}) requires defining, for each n , $2^{N+n} + 1$ functions, which we will call $w_n^{(k)}$, $k = 0, \dots, 2^{N+n}$. However, in contrast with the optimizer for a_f , the definition of $w_n^{(k)}$ now involves not only $w_n^{(k+1)}$, but also $w_n^{(k-1)}$.

Since $s_{2^{N+n}} = 1$, we set

$$(7.9) \quad w_n^{(2^{N+n})}(t) = 1, \quad t \in (0, 1).$$

Assume for the moment that we have already defined the function $w_n^{(0)}$ corresponding to $s_0 = L$. These two “end-point” functions are enough for us to derive $w_n^{(k)}$ for all other k . For all $k = 1, \dots, 2^{N+n} - 1$, we split the point (s_k, s_k) along the line $x_2 = x_1$ (let us call this a *normal split*):

$$(s_k, s_k) = \frac{1}{2}((s_{k-1}, s_{k-1}) + (s_{k+1}, s_{k+1}))$$

and define $w_n^{(k)}$ to be a concatenation of $w_n^{(k-1)}$ and $w_n^{(k+1)}$:

$$(7.10) \quad w_n^{(k)}(t) = \begin{cases} w_n^{(k-1)}(2t), & t \in (0, 1/2) \\ w_n^{(k+1)}(2t - 1), & t \in (1/2, 1). \end{cases}$$

The reader can convince himself that this does define each $w_n^{(k)}$ almost everywhere on $(0, 1)$ up to the knowledge of $w_n^{(0)}$, by writing this construction inductively, as done in a similar setting in [17]. To find $w_n^{(0)}$, we use the tangential split from the previous case. As in (7.6), define $w_n^{(0)}$ through $w_n^{(1)}$ by

$$(7.11) \quad w_n^{(0)}(t) = \begin{cases} \frac{s_1}{L + \Delta_n^*} w_n^{(1)}(2t), & t \in (0, 1/2), \\ \frac{s_1}{L - \Delta_n^*} w_n^{(1)}(2t - 1), & t \in (1/2, 1), \end{cases}$$

where $\Delta_n^* = \sqrt{L^2 - s_1^2}$. This definition closes the loop: the function $w_n^{(0)}$ has been defined almost everywhere on $(0, 1)$ and thus so have the functions $w_n^{(k)}$ for all other k . The construction is pictured in Figure 2. We are now ready to prove our main result.

Lemma 7.3. *The sequence $\{w_n^{(m2^n)}\}$ defined by (7.9), (7.10), and (7.11) is an optimizing sequence for $A_{L,f}$ at $x = (s, s)$, where $s = L - m2^{-N}(L - 1)$. Consequently, if Φ is increasing and h is concave, then*

$$B_{Q,\Phi}(x) \geq A_{L,f}(\sqrt{x_1 x_2}), \quad \forall x \in \Omega_Q.$$

Proof. We will again leave it to the reader to verify (7.1) and (7.2), which in this case is done by induction on the parameter $\ell = n + N$. Let us show (7.3). Let

$$\Sigma_k = \sum_{J \in \mathcal{D}(0,1)} |J| \Phi \left(\langle w_n^{(k)} \rangle_J \langle (w_n^{(k)})^{-1} \rangle_J \right) R_J(w_n^{(k)}).$$

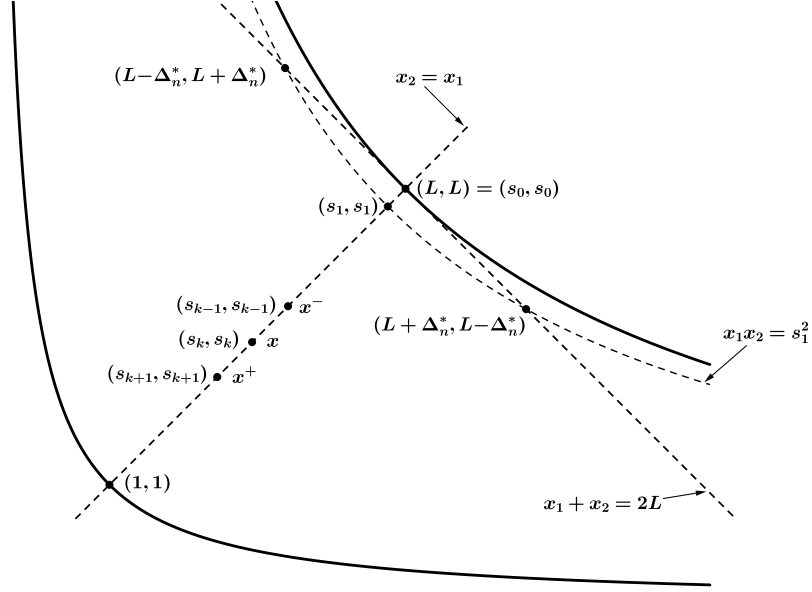


FIGURE 2. The optimizer for $A_{L,f}$: normal splits $x = \frac{1}{2}(x^- + x^+)$ inside the domain; tangential split on the boundary.

Our recursive construction for $w_n^{(k)}$ directly implies that for $k = 1, \dots, 2^{N+n} - 1$,

$$\Sigma_k = \frac{1}{2}\Sigma_{k-1} + \frac{1}{2}\Sigma_{k+1} + 8h(s_k)\Delta_n^2.$$

On the other hand, $\Sigma_{2^{N+n}} = 0$, while (7.11) implies $\Sigma_0 = \Sigma_1 + 8(L^2 - s_1^2)h(L)$. We can easily solve this tridiagonal linear system:

$$\Sigma_k = (2^{N+n} - k)8(L^2 - s_1^2)h(L) + 16\Delta_n^2 \left[\sum_{j=1}^{2^{N+n}-1} (2^{N+n} - j)h(s_j) - \sum_{j=1}^k (k - j)h(s_j) \right],$$

for $k = 0, \dots, 2^{N+n}$ (where for $k = 0$ the second sum in brackets is zero by convention). Since for any i and j , $(i - j)\Delta_n = s_j - s_i$, and $L^2 - s_1^2 = \Delta_n(2L - \Delta_n)$, we have

$$\Sigma_k = 8(s_k - 1)(2L - \Delta_n)h(L) + 16 \left[\sum_{j=1}^{2^{N+n}-1} (s_j - 1)h(s_j)\Delta_n - \sum_{j=1}^k (s_j - s_k)h(s_j)\Delta_n \right].$$

Setting $k = m2^n$ and $s_k = s$, and letting $n \rightarrow \infty$, we get

$$\Sigma_{m2^n} \xrightarrow{n \rightarrow \infty} 16(s - 1)\frac{f(L)}{L} + 16 \left[\int_1^L (z - 1)h(z)dz - \int_s^L (z - s)h(z)dz \right],$$

which is the same thing as $A_{L,f}(s)$ from (4.19).

So far, we have proved that $\mathbf{B}_{Q,\Phi}(x_1, x_2) \geq A_{L,f}(\sqrt{x_1 x_2})$ for all $(x_1, x_2) \in \Omega_Q$ such that $\sqrt{x_1 x_2}$ is a dyadically-rational point in the interval $[1, L]$. Now, any $s \in (1, L)$ can be written as $s = \lim_{n \rightarrow \infty} s_n$, where each s_n is dyadically-rational. Observe that $\mathbf{B}_{Q,\Phi}(s, s)$ is continuous on $(1, L)$ as a function of s , since it is concave on $[1, L]$ by (5.1). Since $A_{L,f}$ is also continuous on $[1, L]$, we can take the limit in the inequality

$$\mathbf{B}_{Q,\Phi}(s_n, s_n) \geq A_{L,f}(s_n),$$

thus completing the proof. \square

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